

<研究ノート>

A Note on Grothendieck Universes

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要旨

この概説では、主に代数幾何学に現れる三角圏や導来圏の性質についてまとめた一連の研究ノートで用いられる集合論的及び圏論的な基礎事項について解説する。第2節において、Zermelo-Fraenkel の公理的集合論の公理系と選択公理について復習する。第3節では、Grothendieck によって圏論に導入された universe の概念に関して、柏原-Schapira による universe の定義を採用し、その性質について解説する。第4節ではさらに、Grothendieck のオリジナルの universe、Bourbaki による universe、MacLane による universe を定義し、それらの公理系を比較する。最終的にそれらがすべて一致することを示す。第5節では、universe の内部において圏論を展開するための基礎事項について解説する。特に、関手圏が小さな圏となることを示す。

1 Introduction

In this exposition, we explain universes in category theory, introduced by A. Grothendieck in SGA4. In section 2, we recall the axioms of ZFC, the Zermelo-Fraenkel set theory with the axiom of choice. In section 3, we introduce the axioms of universes from M. Kashiwara and P. Schapira and discuss their basic properties from the exposition of SGA4. We have the other systems of axioms of universes; Grothendieck's original one and universes of N. Bourbaki and S. MacLane. In section 4, we compare their systems of axioms. We conclude that they give the same systems of axioms of universes, and thus we obtain the same universes. In section 5, we explain the notions of \mathbb{U} -categories and \mathbb{U} -small categories, and we show that the functor category $\mathbf{Fct}(\mathcal{C}, \mathcal{D})$ is also \mathbb{U} -small if two categories \mathcal{C} and \mathcal{D} are \mathbb{U} -small.

2 The Zermelo-Fraenkel Axiomatic Set Theory with the Axiom of Choice

We recall the axioms of the Zermelo-Fraenkel set theory and the axiom of choice. We refer to [Cie97], [Jec03], and [Kun80] for details.

1. Axiom of Extensionality

If X and Y have the same elements, then $X = Y$:

$$\forall u (u \in X \leftrightarrow u \in Y) \rightarrow X = Y.$$

2. Axiom Schema of Separation

Let $\varphi(u, p)$ be a formula. For any X and p , there exists a set $Y = \{u \in X \mid \varphi(u, p)\}$:

$$\forall X \forall p \exists Y \forall u [u \in Y \leftrightarrow u \in X \wedge \varphi(u, p)].$$

By the axiom of extensionality, the set Y is unique.

3. Axiom of Pairing

For any a and b , there exists a set c that contains a and b :

$$\forall a \forall b \exists c \forall x (x = a \vee x = b \rightarrow x \in c).$$

By the axiom schema of separation, we define the set $\{a, b\}$ as

$$\{a, b\} = \{x \in c \mid x = a \vee x = b\}.$$

By the axiom of extensionality, the set $\{a, b\}$ is unique. Then the set $\{a, b\}$ is called the **pair** of a and b . Note that $\{a, b\} = \{b, a\}$. The **singleton** $\{a\}$ is the set $\{a, a\}$.

4. Axiom of Empty Set

There exists the empty set:

$$\exists x \forall y \neg (y \in x).$$

By the axiom of extensionality, the set x is unique. Then we define the symbol \emptyset as

$$\forall a \neg (a \in \emptyset).$$

The set \emptyset is called the **empty set**.

5. Axiom of Union

For every family \mathcal{F} , there exists a set U containing the union of all elements of \mathcal{F} :

$$\forall \mathcal{F} \exists U \forall x [\exists Y (Y \in \mathcal{F} \wedge x \in Y) \rightarrow x \in U].$$

A set of sets is often called a **family** or a **collection** of sets. By the axiom schema of separation, for a family \mathcal{F} of sets, we define the set $\cup \mathcal{F}$ as

$$\cup \mathcal{F} = \{x \in U \mid \exists Y (Y \in \mathcal{F} \wedge x \in Y)\}.$$

By the axiom of extensionality, the set $\cup \mathcal{F}$ is unique. The set $\cup \mathcal{F}$ is called the **union** of \mathcal{F} . We further define $X \cup Y := \cup \{X, Y\}$, the **union** of X and Y .

If $\{X_i \mid i \in I\}$ is a family of sets with the index set I , then the union $\cup \{X_i \mid i \in I\}$ is denoted by $\cup_{i \in I} X_i$.

For a family \mathcal{F} of sets, we define the set $\cap \mathcal{F}$ as

$$\cap \mathcal{F} = \{x \in \cup \mathcal{F} \mid \forall Y (Y \in \mathcal{F} \wedge x \in Y)\}.$$

The set $\cap \mathcal{F}$ is called the **intersection** of \mathcal{F} . We further define $X \cap Y := \cap \{X, Y\}$, the **intersection** of X and Y . If $\{X_i \mid i \in I\}$ is a family of sets with the index set I , then the intersection $\cap \{X_i \mid i \in I\}$ is denoted by $\cap_{i \in I} X_i$.

6. Axiom of Power Set

For every set X , there exists a set P containing the set of all subsets of X :

$$\forall X \exists P \forall u (u \subset X \rightarrow u \in P).$$

We define the symbol \subset as

$$X \subset Y \leftrightarrow \forall u (u \in X \rightarrow u \in Y).$$

Then we say that X is **contained** in Y , and X is called a **subset** of Y .

By the axiom schema of separation, we define the set $\mathcal{P}(X)$ as

$$\mathcal{P}(X) = \{x \in P \mid x \subset X\}.$$

By the axiom of extensionality, the set $\mathcal{P}(X)$ is unique. The set $\mathcal{P}(X)$ is called the **power set** of X .

For every $x \in X$ and $y \in Y$, we define an **ordered pair** (x, y) as

$$(x, y) = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y)).$$

Then we further define a **Cartesian product** $X \times Y$ as

$$X \times Y = \{z \in \mathcal{P}(\mathcal{P}(X \cup Y)) \mid \exists x \in X \exists y \in Y (z = (x, y))\}.$$

7. Axiom of Infinity (Zermelo 1908)

There exists an infinite set:

$$\exists x [\forall z (z = \emptyset \rightarrow z \in x) \wedge \forall y \in x \forall z (z = y \cup \{y\} \rightarrow z \in x)].$$

We say that y is a **successor** of x and write $y = S(x)$ if $S(x) = x \cup \{x\}$.

8. Axiom Schema of Replacement (Fraenkel 1922; Skolem 1922)

For every formula $\varphi(s, t, U, w)$ with free variables s, t, U , and w , every set A , and every parameter p , if $\varphi(s, t, A, p)$ defines a function F on A by

$$F(x) = y \iff \varphi(x, y, A, p),$$

then there exists a set Y containing the range $F[A] = \{F(x) \mid x \in A\}$ of the function F :

$$\forall A \forall p [\forall x \in A \exists! y \varphi(x, y, A, p) \rightarrow \exists Y \forall x \in A \exists y \in Y \varphi(x, y, A, p)],$$

where the quantifier $\exists! x \varphi(x)$ is equivalent to the formula

$$\exists x \varphi(x) \wedge \forall x \forall y [\varphi(x) \wedge \varphi(y) \rightarrow x = y].$$

A subset R of a Cartesian product $X \times Y$ is called a **relation** between X and Y . We usually write aRb instead of $(a, b) \in R$.

A **domain** $\text{dom}(R)$ of a relation is defined as the set of all x such that $(x, y) \in R$ for some $y \in Y$. A **range** $\text{range}(R)$ of a relation is defined as the set of all y such that $(x, y) \in R$ for some $x \in X$.

A relation $R \subset X \times X$ is an **equivalence relation** on X if it is reflexive, symmetric and transitive. The family of all equivalence classes with respect to an equivalence relation R on a set X is called the **quotient set** of X with respect to R and denoted by X/R . Equivalence relations are often denoted by symbol \equiv , and then the quotient set is denoted by X/\equiv .

A relation $R \subset X \times Y$ is called a **function** if

$$\forall x \in X, \forall y_1 \in Y, \forall y_2 \in Y (xRy_1 \wedge xRy_2 \rightarrow y_1 = y_2).$$

For a function f , if $\text{dom}(f) = X$ and $\text{range}(f) \subset Y$, then f is called a **function** (or **map**) from X into Y and it is denoted by $f : X \rightarrow Y$. The set of all functions from X into Y is denoted by Y^X .

If moreover $\text{range}(f) = Y$, then f is said to be a function from X **onto** Y , or a **surjective** function. A function $f : X \rightarrow Y$ is a **one-to-one** (or **injective**) function if

$$f(x) = f(y) \rightarrow x = y$$

for all $x, y \in X$. A function $f : X \rightarrow Y$ is a **bijection**, or a **bijective** function if it is one-to-one and onto Y .

9. Axiom of Regularity (Skolem 1922; von Neumann 1925)

Every nonempty set has an \in -minimal element:

$$\forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y))].$$

10. Axiom of Choice (Levi 1902; Zermelo 1904)

For every family \mathcal{F} of disjoint nonempty sets, there exists a set S that intersects every $x \in \mathcal{F}$ in precisely one point:

$$\begin{aligned} \forall \mathcal{F} [\forall x \in \mathcal{F} (x \neq \emptyset) \wedge \forall x \in \mathcal{F} \forall y \in \mathcal{F} (x = y \vee x \cap y = \emptyset)] \\ \rightarrow \exists S \forall x \in \mathcal{F} \exists! z (z \in S \wedge z \in x). \end{aligned}$$

The system of axioms 0-9 is usually called *Zermelo-Fraenkel set theory* and is abbreviated by ZF. The system of axioms 0-10 is usually denoted by ZFC. Thus, ZFC is the same as ZF+AC, where AC stands for the axiom of choice.

3 Universes

We are assuming the Zermelo-Fraenkel set theory and the axiom of choice for set theory. In this section, we recall the axioms of universes from M. Kashiwara and P. Schapira [KS06] and thier basic properties from SGA 4 [AGV72].

Definition 3.1. A set \mathbb{U} is a **universe** if the following axioms are satisfied:

- (I) if $x \in \mathbb{U}$, then $x \subset \mathbb{U}$;
- (II) if $x \in \mathbb{U}$, then $\{x\} \in \mathbb{U}$;
- (III) if $x \in \mathbb{U}$, then $\mathcal{P}(x) \in \mathbb{U}$;
- (IV) if $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\cup_{i \in I} x_i \in \mathbb{U}$;
- (V) $\mathbb{N} \in \mathbb{U}$. □

We denote the least nonzero limit ordinal ω (or \mathbb{N}). The ordinals less than ω (elements of \mathbb{N}) are called *finite ordinals*, or *natural numbers*. Thus

$$\mathbb{N} = \omega = \{0, 1, 2, \dots\}$$

is the set of all finite ordinals.

We call the above universe the *universe of Kashiwara and Schapira* (or a *Kashiwara-Schapira universe*).

Proposition 3.2. *Let \mathbb{U} be a set satisfying the axioms: (I) and (III). If $x \in \mathbb{U}$ and if $y \subset x$, then $y \in \mathbb{U}$.*

Proof. $\mathcal{P}(x) \subset \mathbb{U}$ by the axioms (III) and (I). Since $y \in \mathcal{P}(x)$, we obtain $y \in \mathbb{U}$. \square

Corollary 3.3. *Let \mathbb{U} be a nonempty set satisfying the axioms: (I) and (III). Then we have $\emptyset \in \mathbb{U}$.*

Proof. A set \mathbb{U} is nonempty, and $\emptyset \subset x$ for any set $x \in \mathbb{U}$. Hence we have $\emptyset \in \mathbb{U}$ by Proposition 3.2. \square

Proposition 3.4. *Let \mathbb{U} be a set satisfying the all the axioms above: (I), (II), (III), (IV) and (V). If $x \in \mathbb{U}$ and $y \in \mathbb{U}$, then $\{x, y\} \in \mathbb{U}$.*

Proof. Put $x_1 := x$ and $x_2 := y$. We have $\{x_1\} \in \mathbb{U}$ and $\{x_2\} \in \mathbb{U}$ by the axiom (II). Since $\mathbb{N} \in \mathbb{U}$ by the axiom (V), $\{1, 2\} \in \mathbb{U}$ by Proposition 3.2. Hence we obtain $\{x, y\} = \{x_1\} \cup \{x_2\} \in \mathbb{U}$ by the axiom (IV). \square

Proposition 3.5. *Let \mathbb{U} be a set satisfying the axiom (I). If $\{x, y\} \in \mathbb{U}$, then $x \in \mathbb{U}$ and $y \in \mathbb{U}$.*

Proof. We have $\{x, y\} \subset \mathbb{U}$ by the axiom (I). Hence we obtain $x \in \mathbb{U}$ and $y \in \mathbb{U}$. \square

Proposition 3.6. *Let \mathbb{U} be a set satisfying all the axioms above: (I), (II), (III), (IV) and (V). If $x \in \mathbb{U}$ and $y \in \mathbb{U}$, then $(x, y) \in \mathbb{U}$.*

Proof. We have $\{x\} \in \mathbb{U}$ and $\{x, y\} \in \mathbb{U}$ by the axiom (II) and Proposition 3.4. Hence we obtain $(x, y) \in \mathbb{U}$ by Proposition 3.4 again. \square

Proposition 3.7. *Let \mathbb{U} be a set satisfying the axiom (I). If $(x, y) \in \mathbb{U}$, then $x \in \mathbb{U}$ and $y \in \mathbb{U}$.*

Proof. By definition, $(x, y) = \{\{x\}, \{x, y\}\}$. Hence this follows from Proposition 3.5. \square

Corollary 3.8. *Let \mathbb{U} be a set satisfying all the axioms above: (I), (II), (III), (IV) and (V). If $x_i \in \mathbb{U}$ for each $i = 1, 2, \dots, n$, then $(x_1, x_2, \dots, x_n) \in \mathbb{U}$.*

Proof. By definition, $(x_1, x_2, \dots, x_n) = ((x_1, x_2, \dots, x_{n-1}), x_n)$. Hence we obtain $(x_1, x_2, \dots, x_n) \in \mathbb{U}$ inductively. \square

Proposition 3.9. *Let \mathbb{U} be a set satisfying the axioms: (I) and (III). If $x \in \mathbb{U}$ and y is a quotient set of x by some equivalence relation, then $y \in \mathbb{U}$.*

Proof. Since the quotient set y is a set of the equivalence classes and the equivalence classes are subsets of x , y is a subset of $\mathcal{P}(x)$. Hence we obtain $y \in \mathbb{U}$ by the axiom (III) and Proposition 3.2. \square

Proposition 3.10. *Let \mathbb{U} be a set satisfying the axioms: (II) and (IV). If $X \in \mathbb{U}$, $Y \subset \mathbb{U}$, and $f : X \rightarrow Y$ is a surjective function, then $Y \in \mathbb{U}$.*

Proof. Since $f(x) \in Y$, we have $f(x) \in \mathbb{U}$, and $\{f(x)\} \in \mathbb{U}$ for all $x \in X$ by the axiom (II). Hence we obtain $Y = \cup_{x \in X} \{f(x)\} \in \mathbb{U}$ by the axiom (IV). \square

Corollary 3.11. *Let \mathbb{U} be a set satisfying the axioms: (I) and (IV). If $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\{x_i \mid i \in I\} \in \mathbb{U}$.*

Proof. Since $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, we have $\{x_i \mid i \in I\} \subset \mathbb{U}$. The function $f : I \rightarrow \{x_i \mid i \in I\}$ defined by $f(i) = x_i$ is surjective. Hence we obtain $\{x_i \mid i \in I\} \in \mathbb{U}$ by Proposition 3.10. \square

Proposition 3.12. *Let \mathbb{U} be a set satisfying the axioms: (I), (III), and (IV). If $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\cap_{i \in I} x_i \in \mathbb{U}$.*

Proof. Since $\cap_{i \in I} x_i \subset \cup_{i \in I} x_i$, we have $\cap_{i \in I} x_i \in \mathbb{U}$ by the axiom (IV) and Proposition 3.2. \square

Proposition 3.13. *Let \mathbb{U} be a set satisfying all the axioms above: (I), (II), (III), (IV) and (V). If $X \in \mathbb{U}$ and $Y \in \mathbb{U}$, then $X \times Y \in \mathbb{U}$.*

Proof. If $x \in X$ and $y \in Y$, then we have $x \in \mathbb{U}$ and $y \in \mathbb{U}$ by the axiom (I), and $\{(x, y)\} \in \mathbb{U}$ by Proposition 3.6 and the axiom (II). Since $\{x\} \times Y = \cup_{y \in Y} \{(x, y)\}$, we see that $\{x\} \times Y \in \mathbb{U}$ from the axiom (IV). Moreover, since $X \times Y = \cup_{x \in X} (\{x\} \times Y)$, we obtain $X \times Y \in \mathbb{U}$ by the axiom (IV) again. \square

Corollary 3.14. *Let \mathbb{U} be a set satisfying all the axioms above: (I), (II), (III), (IV) and (V). Then we have $\mathbb{Z} \in \mathbb{U}$.*

Proof. By definition, we have

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N}) / \equiv,$$

where \equiv is the equivalence relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$(m, n) \equiv (m', n') \iff m + n' = m' + n.$$

Since $\mathbb{N} \times \mathbb{N} \in \mathbb{U}$ by the axiom (V) and Proposition 3.13, it follows from Proposition 3.9 that $\mathbb{Z} \in \mathbb{U}$. \square

If $\{x_i \mid i \in I\}$ is a family of sets with the index set I , then the **disjoint union** of the family $\{x_i \mid i \in I\}$, denoted by $\bigsqcup_{i \in I} x_i$, is defined as the union of the family of sets $\{x_i \times \{i\} \mid i \in I\}$:

$$\bigsqcup_{i \in I} x_i := \bigcup_{i \in I} (x_i \times \{i\}).$$

Corollary 3.15. *Let \mathbb{U} be a set satisfying all the axioms above: (I), (II), (III), (IV) and (V).*

If $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\bigsqcup_{i \in I} x_i \in \mathbb{U}$.

Proof. For any $i \in I$, we have $\{i\} \in \mathcal{P}(I)$. Then we obtain $\{i\} \in \mathbb{U}$, since $\mathcal{P}(I) \subset \mathbb{U}$ by the axioms (III) and (I). Hence we have $\bigsqcup_{i \in I} x_i \in \mathbb{U}$ by Proposition 3.13 and the axiom (IV). \square

Proposition 3.16. *Let \mathbb{U} be a set satisfying the axioms: (I), (II), (III) and (IV). If $X \in \mathbb{U}$, $Y \in \mathbb{U}$, and R is a relation between X and Y , then $R \in \mathbb{U}$. In particular, any function from X into Y is an element of \mathbb{U} .*

Proof. We have $R \in \mathbb{U}$ by Proposition 3.13 and Proposition 3.2. \square

Proposition 3.17. *Let \mathbb{U} be a set satisfying the axioms: (I), (II), (III) and (IV). If $X \in \mathbb{U}$ and $Y \in \mathbb{U}$, then the set of all relations between X and Y is an element of \mathbb{U} . In particular, $Y^X \in \mathbb{U}$.*

Proof. Let Z be the set of all relations between X and Y . Then Z is a subset of $\mathcal{P}(X \times Y) \times \{X\} \times \{Y\}$. We have $\mathcal{P}(X \times Y) \times \{X\} \times \{Y\} \in \mathbb{U}$ by Proposition 3.13 and the axioms (II) and (III). Hence we obtain $Z \in \mathbb{U}$ by Proposition 3.2. \square

If $\{x_i \mid i \in I\}$ is a family of sets with the index set I , then the **Cartesian product** of the family $\{x_i \mid i \in I\}$, denoted by $\prod_{i \in I} x_i$, is defined as follows:

$$\prod_{i \in I} x_i := \{f \in (\cup_{i \in I} x_i)^I \mid f(i) \in x_i \text{ for each } i \in I\}.$$

Note that the Cartesian product is a subset of $(\cup_{i \in I} x_i)^I$.

Corollary 3.18. *Let \mathbb{U} be a set satisfying the axioms: (I), (II), (III) and (IV). If $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\prod_{i \in I} x_i \in \mathbb{U}$.*

Proof. It follows from the definition that $\prod_{i \in I} x_i \subset (\cup_{i \in I} x_i)^I$. Since $\cup_{i \in I} x_i \in \mathbb{U}$ by the axiom (IV), we have $(\cup_{i \in I} x_i)^I \in \mathbb{U}$ by Proposition 3.17. Hence we obtain $\prod_{i \in I} x_i \in \mathbb{U}$ by Proposition 3.2. \square

For a set X , the **cardinality** of X is denoted by $|X|$.

Theorem 3.19. *Let X and Y be any sets. Then the following conditions are equivalent:*

1. $|X| \leq |Y|$;
2. *there exists an injective function $\varphi : X \rightarrow Y$.*

Moreover, if $X \neq \emptyset$, then these conditions are equivalent to the condition that

3. *there exists a surjective function $\psi : Y \rightarrow X$.*

Proof. For example, see [Cie97, Theorem 5.1.2]. \square

Proposition 3.20. *Let \mathbb{U} be a set satisfying the axioms: (II) and (IV). If $X \subset \mathbb{U}$ and $|X| \leq |Y|$ for some $Y \in \mathbb{U}$, then $X \in \mathbb{U}$.*

Proof. We may assume that $X \neq \emptyset$. It follows from Theorem 3.19 that there exists a surjective map $\psi : Y \rightarrow X$. Hence we obtain $X \in \mathbb{U}$ by Proposition 3.10. \square

Proposition 3.21. *Let \mathbb{U} be a set satisfying the axioms: (I) and (III). If $x \in \mathbb{U}$, then $|x| \not\leq |\mathbb{U}|$. In particular, $\mathbb{U} \notin \mathbb{U}$.*

Proof. We have $|x| \leq |\mathbb{U}|$ from the axiom (I). Suppose that $|x| = |\mathbb{U}|$. Then we obtain $|\mathbb{U}| < |\mathcal{P}(x)|$, because $|x| < |\mathcal{P}(x)|$ by Cantor's theorem (see [Cie97, Theorem 5.1.6]). On the other hand, we have $\mathcal{P}(x) \subset \mathbb{U}$ by the axioms (I) and (III). Hence $|\mathcal{P}(x)| \leq |\mathbb{U}|$. This is a contradiction. \square

Proposition 3.22. *If $(\mathbb{U}_\lambda)_{\lambda \in \Lambda}$ is a nonempty family of universes, then $\bigcap_{\lambda \in \Lambda} \mathbb{U}_\lambda$ is a universe.*

Proof. This follows from the definition. \square

4 Comparison of Axioms of Universes

We have the other axioms of universes of A. Grothendieck [Gab62], N. Bourbaki [AGV72], and S. MacLane [Mac88].

Definition 4.1. A set \mathbb{U} is a **universe of Grothendieck** (or a **Grothendieck universe**) if

the following axioms are satisfied:

- (G.1) if $x \in \mathbb{U}$, then $x \subset \mathbb{U}$;
- (G.2) if $x \in \mathbb{U}$, then $\{x\} \in \mathbb{U}$;
- (G.3) if $x \in \mathbb{U}$, then $\mathcal{P}(x) \in \mathbb{U}$;
- (G.4) if $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\cup_{i \in I} x_i \in \mathbb{U}$;
- (G.5) $(x, y) \in \mathbb{U}$ if and only if $x \in \mathbb{U}$ and $y \in \mathbb{U}$;
- (G.6) $\mathbb{Z} \in \mathbb{U}$. □

Definition 4.2. A set \mathbb{U} is a **universe of Bourbaki** (or a **Bourbaki universe**) if the following axioms are satisfied:

- (B.1) if $x \in \mathbb{U}$, then $x \subset \mathbb{U}$;
- (B.2) if $x \in \mathbb{U}$ and $y \in \mathbb{U}$, then $\{x, y\} \in \mathbb{U}$;
- (B.3) if $x \in \mathbb{U}$, then $\mathcal{P}(x) \in \mathbb{U}$;
- (B.4) if $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\cup_{i \in I} x_i \in \mathbb{U}$;
- (B.5) $\mathbb{U} \neq \emptyset$. □

Definition 4.3. A set \mathbb{U} is a **universe of MacLane** (or a **MacLane universe**) if the following axioms are satisfied:

- (M.1) if $x \in \mathbb{U}$, then $x \subset \mathbb{U}$;
- (M.2) if $x \in \mathbb{U}$ and $y \in \mathbb{U}$, then $\{x, y\} \in \mathbb{U}$;
- (M.3) if $x \in \mathbb{U}$ and $y \in \mathbb{U}$, then $(x, y) \in \mathbb{U}$;
- (M.4) if $x \in \mathbb{U}$ and $y \in \mathbb{U}$, then $x \times y \in \mathbb{U}$;
- (M.5) if $x \in \mathbb{U}$, then $\mathcal{P}(x) \in \mathbb{U}$;
- (M.6) if $x \in \mathbb{U}$, then $\cup x \in \mathbb{U}$;
- (M.7) $\omega \in \mathbb{U}$;
- (M.8) if $x \in \mathbb{U}$, $y \subset \mathbb{U}$ and $f: x \rightarrow y$ is a surjective function, then $y \in \mathbb{U}$. □

Note that

- (I) = (G.1) = (B.1) = (M.1);
- (II) = (G.2);
- (III) = (G.3) = (B.3) = (M.5);
- (IV) = (G.4) = (B.4);
- (V) = (M.7);
- (B.2) = (M.2).

4.1 Comparison of universes of Kashiwara-Schapira and of Grothendieck

Proposition 4.4. *Kashiwara-Schapira universes satisfy the axiom (G.5).*

Proof. This follows from Propsitions 3.6 and 3.7. □

Proposition 4.5. *Kashiwara-Schapira universes satisfy the axiom (G.6).*

Proof. This follows from Corollary 3.14. □

Proposition 4.6. *Grothendieck universes satisfy the axiom (V).*

Proof. Let \mathbb{U} be a Grothendieck universe. By Proposition 3.2, we have $\mathbb{N} \in \mathbb{U}$, since $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{Z} \in \mathbb{U}$ by the axiom (G.6). \square

Theorem 4.7. *The universe of Kashiwara-Schapira is the same as the universe of Grothendieck.* \square

4.2 Comparison of universes of Kashiwara-Schapira and of Bourbaki

Proposition 4.8. *Kashiwara-Schapira universes satisfy the axiom (B.2).*

Proof. This follows from Proposition 3.4. \square

Proposition 4.9. *Kashiwara-Schapira universes satisfy the axiom (B.5).*

Proof. This follows from the axiom (V). \square

Proposition 4.10. *Bourbaki universes satisfy the axiom (II).*

Proof. This follows from the definition of singletons and the axiom (B.2). \square

Proposition 4.11. *Bourbaki universes satisfy the axiom (V).*

Proof. Let \mathbb{U} be a Bourbaki universe. By Corollary 3.3, we have $\emptyset \in \mathbb{U}$. Then we obtain $\mathbb{N} \in \mathbb{U}$ by Proposition 4.10 and the axiom (B.2). \square

Theorem 4.12. *The universe of Kashiwara-Schapira is the same as the universe of Bourbaki.* \square

4.3 Comparison of universes of Kashiwara-Schapira and of MacLane

Proposition 4.13. *Kashiwara-Schapira universes satisfy the axiom (M.2).*

Proof. This follows from Proposition 4.8. \square

Proposition 4.14. *Kashiwara-Schapira universes satisfy the axiom (M.3).*

Proof. This follows from Proposition 3.13. \square

Proposition 4.15. *Kashiwara-Schapira universes satisfy the axiom (M.4).*

Proof. This follows from Proposition 3.6. \square

Proposition 4.16. *Kashiwara-Schapira universes satisfy the axiom (M.6).*

Proof. Let \mathbb{U} be a Kashiwara-Schapira universe. If $x \in \mathbb{U}$, we have $y \in \mathbb{U}$ for every $y \in x$ by the axiom (I). Then we obtain $\cup x = \cup_{y \in x} y \in \mathbb{U}$ by the axiom (IV). \square

Proposition 4.17. *Kashiwara-Schapira universes satisfy the axiom (M.8).*

Proof. This follows from Proposition 3.10. \square

Proposition 4.18. *MacLane universes satisfy the axiom (II).*

Proof. This follows from the definition of singletons and the axiom (M.2). \square

Proposition 4.19. *MacLane universes satisfy the axiom (IV).*

Proof. Let \mathbb{U} be a MacLane universe. If $I \in \mathbb{U}$ and $x_i \in \mathbb{U}$ for all $i \in I$, then $\{x_i \mid i \in I\} \subset \mathbb{U}$. We have a surjective function $f : I \rightarrow \{x_i \mid i \in I\}$ defined by $f(i) = x_i$. Hence we obtain $\{x_i \mid i \in I\} \in \mathbb{U}$ by the axiom (M.8). Therefore $\cup_{i \in I} x_i = \cup \{x_i \mid i \in I\} = \cup x \in \mathbb{U}$. \square

Theorem 4.20. *The universe of Kashiwara-Schapira is the same as the universe of MacLane.* \square

5 Categories and Functors in Universes

In this section, we introduce basic notions of categories and functors in universes to fix some notations needed for the paper [OT14]. We refer to [Mac88], [Sch72], [KS06], and [Yek20] for details.

In this section, we assume that \mathbb{U} is a (fixed) universe. The notions of categories and functors are defined in [OT14, Section 2].

5.1 \mathbb{U} -categories

A set is called a \mathbb{U} -small (or **small**) set if it is an element of \mathbb{U} . Thus the universe \mathbb{U} is the set of all \mathbb{U} -small sets, but \mathbb{U} itself is not a \mathbb{U} -small set by Proposition 3.21.

We define a \mathbb{U} -class (or a **class**) C to be any subset C of the universe \mathbb{U} . It follows from the axiom (I) that every \mathbb{U} -small set is also a \mathbb{U} -class. A \mathbb{U} -class C is called a **proper class** if it is not a \mathbb{U} -small set. In particular, the universe \mathbb{U} itself is a proper class.

A category \mathcal{C} is called a \mathbb{U} -category if the set $\text{Ob}(\mathcal{C})$ of objects is a \mathbb{U} -class and the set $\text{Hom}_{\mathcal{C}}(X, Y)$ of morphisms is a \mathbb{U} -small set for any $X, Y \in \text{Ob}(\mathcal{C})$. A \mathbb{U} -category \mathcal{C} is said to be a \mathbb{U} -small category if $\text{Ob}(\mathcal{C})$ is a \mathbb{U} -small set.

We denote the category of \mathbb{U} -small sets and maps by $\mathbb{U}\text{-Set}$. $\mathbb{U}\text{-Set}$ is a \mathbb{U} -category, since $\text{Ob}(\mathbb{U}\text{-Set}) = \mathbb{U}$ and $\text{Hom}_{\mathbb{U}\text{-Set}}(X, Y) \in \mathbb{U}$ for any $X, Y \in \mathbb{U}$ by Propositions 3.17 and 3.2.

5.2 Functor categories

For a category \mathcal{C} , we denote the set of morphisms of \mathcal{C} by $\text{Mor}(\mathcal{C})$:

$$\text{Mor}(\mathcal{C}) = \bigcup_{(X,Y) \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(X, Y).$$

Furthermore, we denote the set of identity morphisms of \mathcal{C} by $1\text{-Mor}(\mathcal{C})$. Then we have the isomorphism

$$\text{Ob}(\mathcal{C}) \xrightarrow{\sim} 1\text{-Mor}(\mathcal{C}), \quad X \mapsto \text{id}_X.$$

Lemma 5.1. *If \mathcal{C} is a \mathbb{U} -small category, then $\text{Mor}(\mathcal{C})$ is a \mathbb{U} -small set. In particular, $1\text{-Mor}(\mathcal{C})$ is a \mathbb{U} -small set.*

Proof. The index set $\text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})$ is a \mathbb{U} -small set by Proposition 3.13. Then $\text{Mor}(\mathcal{C})$ is a \mathbb{U} -small set by the axiom (IV). \square

Lemma 5.2. *If \mathcal{C} is a \mathbb{U} -small category, then*

$$\prod_{(X,Y) \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(X, Y)$$

is a \mathbb{U} -small set.

Proof. The index set $\text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})$ is a \mathbb{U} -small set by Proposition 3.13. Then we see

that $\prod_{(X,Y) \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(X, Y)$ is a \mathbb{U} -small set by Corollary 3.18. \square

Consider a family $\{\mathcal{C}_i\}_{i \in I}$ of categories indexed by a set I . Then we define the **product category** $\prod_{i \in I} \mathcal{C}_i$ by setting:

$$\begin{aligned} \text{Ob}\left(\prod_{i \in I} \mathcal{C}_i\right) &= \prod_{i \in I} \text{Ob}(\mathcal{C}_i), \\ \text{Hom}_{\prod_{i \in I} \mathcal{C}_i}(\{X_i\}_{i \in I}, \{Y_i\}_{i \in I}) &= \prod_{i \in I} \text{Hom}_{\mathcal{C}_i}(X_i, Y_i). \end{aligned}$$

For two categories \mathcal{C}, \mathcal{D} , the product category of \mathcal{C} and \mathcal{D} is denoted by $\mathcal{C} \times \mathcal{D}$.

Proposition 5.3. *If \mathcal{C} and \mathcal{D} are \mathbb{U} -small categories, then the product category $\mathcal{C} \times \mathcal{D}$ is a \mathbb{U} -small category.*

Proof. The set of objects $\text{Ob}(\mathcal{C} \times \mathcal{D}) = \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$ and the set of morphisms

$$\text{Hom}_{\mathcal{C} \times \mathcal{D}}((X_1, Y_1), (X_2, Y_2)) = \text{Hom}_{\mathcal{C}}(X_1, X_2) \times \text{Hom}_{\mathcal{D}}(Y_1, Y_2)$$

are \mathbb{U} -small sets by Proposition 3.13. \square

Proposition 5.4. *If \mathcal{C} is a \mathbb{U} -small category and \mathcal{D} is a \mathbb{U} -category, then the functor category $\text{Fct}(\mathcal{C}, \mathcal{D})$ is a \mathbb{U} -category. Moreover, if \mathcal{D} is a \mathbb{U} -small category, then $\text{Fct}(\mathcal{C}, \mathcal{D})$ is a \mathbb{U} -small category.*

Proof. If \mathcal{C} is the empty category, then $\text{Fct}(\mathcal{C}, \mathcal{D})$ has exactly one element, (that is, the empty functor) and its identity morphism. If \mathcal{D} is the empty category but \mathcal{C} is not empty, then $\text{Fct}(\mathcal{C}, \mathcal{D})$ is empty.

Now we assume that \mathcal{C} and \mathcal{D} are not empty. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. The set of morphisms from F to G , $\text{Hom}_{\text{Fct}(\mathcal{C}, \mathcal{D})}(F, G)$, is a subset of

$$\text{Mor}(F(\mathcal{C})) \times \text{Mor}(G(\mathcal{C})) \times \prod_{X \in \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{D}}(F(X), G(X)),$$

where we put

$$\text{Mor}(F(\mathcal{C})) = \bigcup_{(X,Y) \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

and

$$\text{Mor}(G(\mathcal{C})) = \bigcup_{(X,Y) \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{D}}(G(X), G(Y)).$$

If \mathcal{C} is a \mathbb{U} -small category and \mathcal{D} is a \mathbb{U} -category, then $\text{Mor}(F(\mathcal{C}))$, $\text{Mor}(G(\mathcal{C}))$, and

$$\prod_{X \in \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{D}}(F(X), G(X))$$

are \mathbb{U} -small sets by the axiom (IV) and Corollary 3.18. Hence we see that $\text{Hom}_{\text{Fct}(\mathcal{C}, \mathcal{D})}(F, G)$ is also a \mathbb{U} -small set by Propositions 3.13 and 3.2. Therefore $\text{Fct}(\mathcal{C}, \mathcal{D})$ is a \mathbb{U} -category.

Moreover, we assume that \mathcal{D} is a \mathbb{U} -small category. Then $\text{Ob}(\text{Fct}(\mathcal{C}, \mathcal{D}))$ is a \mathbb{U} -small set by Lemma 5.1, Lemma 5.2, and Propositions 3.17 and 3.2, since

$$\text{Ob}(\text{Fct}(\mathcal{C}, \mathcal{D})) \subset \text{Hom}_{\mathbb{U}\text{-Set}}(\text{Mor}(\mathcal{C}), \text{Mor}(\mathcal{D})).$$

Therefore $\text{Fct}(\mathcal{C}, \mathcal{D})$ is a \mathbb{U} -small category. \square

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