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＜研究ノート＞

Triangulated Categories V:

Glueing t-structures

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要旨

この概説は，主に代数幾何学に現れる三角圏や導来圏の性質についてまとめた一連の研究ノートの第５部であり，今回は t–構造の貼り合わせに関して解説し，偏屈 t–構造を定義する．第２節において t–構造を保存する関手として t–完全関手の定義と基本性質を，第３節において t–構造の貼り合わせの構成について解説する．第４節では貼り合わせ条件について詳しく調べ，偏屈 t–構造を定義する．また，代数幾何学からの例として双有理幾何学における偏屈連接層の定義について述べる．

1 Introduction

This exposition is the fifth part of our study of triangulated categories in algebraic geometry. We shall explain how to glue t-structures on triangulated categories and describe their proofs in detail. Main reference is [BBD].

2 t-exact functors

Definition 2.1. Let $\mathcal{C}$, $\mathcal{D}$ be triangulated categories and let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor. Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be t-structures on $\mathcal{C}$, $\mathcal{D}$, respectively. We say that F is t-**exact** if $F\mathcal{C}^{\geq 0} \subset \mathcal{D}^{\geq 0}$ and $F\mathcal{C}^{\leq 0} \subset \mathcal{D}^{\leq 0}$. In the case, the t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ are said to be **compatible.**

Proposition 2.2. *Let $\mathcal{C}$, $\mathcal{D}$ and F be as in Definition 2.1. Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\leq 0})$, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\leq 0})$ be t-structures on $\mathcal{C}$, $\mathcal{D}$, respectively. Let $\tau^{\mathcal{C}}_{\leq n}$, $\tau^{\mathcal{C}}_{\geq n}$ (resp. $\tau^{\mathcal{D}}_{\leq n}$, $\tau^{\mathcal{D}}_{\geq n}$) be truncation functors for each $n \in \mathbb{Z}$ with respect to the t-structure of $\mathcal{C}$ (resp. $\mathcal{D}$). If $F : \mathcal{C} \to \mathcal{D}$ is t-exact, then we have*

$$F \circ \tau^{\mathcal{C}}_{\leq n} \simeq \tau^{\mathcal{D}}_{\leq n} \circ F \quad \text{and} \quad F \circ \tau^{\mathcal{C}}_{\geq n} \simeq \tau^{\mathcal{D}}_{\geq n} \circ F$$

for each $n \in \mathbb{Z}$.

Proof. Let X be any object of $\mathcal{C}$. For any $n \in \mathbb{Z}$, we have the distinguished triangles

$$\tau_{\leq n} X \to X \to \tau_{\geq n+1} X \to T\tau_{\leq n} X$$

and

$$F(\tau_{\leq n} X) \to F(X) \to F(\tau_{\geq n+1} X) \to TF(\tau_{\leq n} X).$$

By Proposition 2.9 of [Ter 15], we obtain $F(\tau_{\leq n}X) \simeq \tau_{\leq n}F(X)$ and $F(\tau_{\geq n}X) \simeq \tau_{\geq n}F(X)$. □

Proposition 2.3. *Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be t−structures on triangulated categories $\mathcal{C}$ and D, respectively. Let $F: \mathcal{C} \to \mathcal{D}$ be a fully faithful exact functor. Assume that F is a t−exact functor. Then we have the following.*

(*i*) *If $F(X) \in \mathcal{D}^{\leq 0}$ for $X \in \mathcal{C}$, then $X \in \mathcal{C}^{\leq 0}$, i.e. $F\mathcal{C}^{\leq 0} = F\mathcal{C} \cap \mathcal{D}^{\leq 0}$.*
(*ii*) *If $F(X) \in \mathcal{D}^{\geq 0}$ for $X \in \mathcal{C}$, then $X \in \mathcal{C}^{\geq 0}$, i.e. $F\mathcal{C}^{\geq 0} = F\mathcal{C} \cap \mathcal{D}^{\geq 0}$.*

Proof. (*i*) If $F(X) \in \mathcal{D}^{\leq 0}$ for $X \in \mathcal{C}$, then for any $Y \in \mathcal{C}^{>0}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) = 0$ since $F(Y) \in \mathcal{D}^{\geq 0}$. Hence we have $X \in {}^{\perp}(\mathcal{C}^{>0}) = \mathcal{C}^{\geq 0}$ by Proposition 2.6 of [Ter 15].
(*ii*) If $F(X) \in \mathcal{D}^{\geq 0}$ for $X \in \mathcal{C}$, then for any $Y \in \mathcal{C}^{<0}$ $\mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X)) = 0$ since $F(Y) \in \mathcal{D}^{<0}$. Hence we have $X \in (\mathcal{C}^{<0})^{\perp} = \mathcal{C}^{\geq 0}$ by Proposition 2.6 of [Ter 15]. □

Corollary 2.4. *Let $\mathcal{D}$ be a triangulated category and let $\mathcal{C}$ be full triangulated subcategory of $\mathcal{D}$. Assume that there exist t−structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{C}$, $\mathcal{D}$, respectively. If the inclusion functor $\tau: \mathcal{C} \to \mathcal{D}$ is t−exact, then a t−structure on $\mathcal{D}$ determines a unique compatible t−structure on $\mathcal{C}$.* □

Proposition 2.5. *Let $(\mathcal{D}, T)$ be a triangulated category and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t−structure on $\mathcal{D}$. Assume that $\mathcal{C}$ is a strictly full triangulated subcategory of $\mathcal{D}$. Then the pair*

$$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) := (\mathcal{C} \cap \mathcal{D}^{\leq 0}, \mathcal{C} \cap \mathcal{D}^{\geq 0})$$

*is a t−structure on $\mathcal{C}$ if and only if $\mathcal{C}$ is stable under the functor $\tau^{\mathcal{D}}_{\leq 0}$, i.e. $\tau^{\mathcal{D}}_{\leq 0}\mathcal{C} \subset \mathcal{C}$. In this case, the t−structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ on $\mathcal{C}$ is called the **induced t−structure**.*

Proof. If $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a *t*-structure on $\mathcal{C}$, then τ is *t*-exact. By Proposition 2.2, we obtain $\tau^{\mathcal{D}}_{\leq 0} X \simeq \tau^{\mathcal{C}}_{\leq 0} X \in \mathcal{C}^{\leq 0}$ for any $X \in \mathcal{C}$. Hence $\tau^{\mathcal{D}}_{\leq 0}\mathcal{C} \subset \mathcal{C}$.

Conversely, assume that $\tau^{\mathcal{D}}_{\leq 0}\mathcal{C} \subset \mathcal{C}$. Let us check the conditions of Definition 2.1 of [OT 14] to show that the pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a *t*-structure on $\mathcal{C}$.

(*a*) We have

$$T\mathcal{C}^{\leq 0} = T(\mathcal{C} \cap \mathcal{D}^{\geq 0}) \subset \mathcal{C} \cap T\mathcal{D}^{\leq 0} \subset \mathcal{C} \cap \mathcal{D}^{\leq 0} = \mathcal{C}^{\leq 0}$$

and

$$T^{-1}\mathcal{C}^{\geq 0} = T^{-1}(\mathcal{C} \cap \mathcal{D}^{\geq 0}) \subset \mathcal{C} \cap T^{-1}\mathcal{D}^{\geq 0} \subset \mathcal{C} \cap \mathcal{D}^{\geq 0} = \mathcal{C}^{\geq 0}.$$

(*b*) is clear because $\mathcal{C}^{\leq 0} \subset \mathcal{D}^{\leq 0}$ and $T^{-1}\mathcal{C}^{\geq 0} \subset T^{-1}\mathcal{D}^{\geq 0}$.

(c) For each $X \in \mathcal{C}$, we have a distinguished triangle

$$\tau^{\mathcal{D}}_{\leq 0} X \to X \to \tau^{\mathcal{D}}_{\geq 1} X \to T\tau^{\mathcal{D}}_{\leq 0} X$$

in $\mathcal{D}$. If $\tau^{\mathcal{D}}_{\leq 0}\mathcal{C} \subset \mathcal{C}$, then $\tau^{\mathcal{D}}_{\leq 0} X \in \mathcal{C} \cap \mathcal{D}^{\leq 0} = \mathcal{C}^{\leq 0}$. By Corollary 3.19 of [OT 14], we see that $\tau^{\mathcal{D}}_{\geq 1} X$ is isomorphic to an object of $\mathcal{C}$, and thus it belongs to $\mathcal{C}$, because $\mathcal{C}$ is a strictly full subcategory of $\mathcal{D}$. Hence $\tau^{\mathcal{D}}_{\geq 1} X \in \mathcal{C} \cap \mathcal{D}^{\geq 1} = \mathcal{C}^{\geq 1}$. Therefore by Corollary 3.18 of [OT 14], we obtain a decomposition in $\mathcal{C}$. □

3 Gluing t-structures

Let $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories and let $F\colon \mathcal{A} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{B}$ be exact functors. We suppose the following conditions:

GL 1 F admits a left adjoint functor F_L and a right adjoint functor F_R : $F_L \dashv F \dashv F_R$.

GL 2 G admits a left adjoint functor G_L and a right adjoint functor G_R : $G_L \dashv G \dashv G_R$.

GL 3 $G \circ F = 0$.

GL 4 There exist the following two distinguished triangles:

(3.1) $$G_L GX \to X \to F F_L X \xrightarrow{d} TG_L GX,$$

(3.2) $$FF_R X \to X \to G_R GX \xrightarrow{d'} TFF_R X.$$

GL 5 F, G_L and G_R are fully faithful.

Proposition 3.1. *From the above conditions, we obtain the following.*

(i) $F_L \circ G_L = 0$ *and* $F_R \circ G_R = 0$.

(ii) *The morphisms d and d' are unique.*

(iii) *The units and counits of the adjunctions are isomorphisms:*

$$F_L \circ F \xrightarrow{\sim} \mathrm{id}_{\mathcal{A}} \xrightarrow{\sim} F_R \circ F \quad \textit{and} \quad G \circ G_R \xrightarrow{\sim} \mathrm{id}_{\mathcal{B}} \xrightarrow{\sim} G \circ G_L .$$

Proof. (i) For any $Y \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{A}}(F_L G_L X, Y) \simeq \mathrm{Hom}_{\mathcal{D}}(G_L X, FY) \simeq \mathrm{Hom}_{\mathcal{B}}(X, GFY) = 0.$$

For any $X \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{A}}(X, F_R G_R Y) \simeq \mathrm{Hom}_{\mathcal{D}}(FX, G_R Y) \simeq \mathrm{Hom}_{\mathcal{B}}(GFX, Y) = 0.$$

By the Yoneda Lemma, we obtain $F_L \circ G_L = 0$ and $F_R \circ G_R = 0$.

(ii) In the distinguished triangle (3.1), we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(TG_L GX, FF_L X) &\simeq \mathrm{Hom}_{\mathcal{D}}(G_L TGX, FF_L X) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(TGX, GFF_L X) \\ &= 0. \end{aligned}$$

In the distinguished triangle (3.2), we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(TFF_R X, G_R GX) &\simeq \mathrm{Hom}_{\mathcal{D}}(FTF_R, G_R GX) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(GFTF_R X, GX) \\ &= 0. \end{aligned}$$

By Lemma 2.8 of [Ter 15], we see that d and d' are unique.

(iii) This follows from Proposition 2.8 of [Ter 16]. □

Theorem 3.2. *Let* $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ *be a t−structure on* $\mathcal{A}$ *and let* $(\mathcal{B}^{\leq 0}, \mathcal{B}^{\geq 0})$ *be a t−structure on* $\mathcal{B}$. *We define the strictly full subcategories* $\mathcal{D}^{\leq 0}$ *and* $\mathcal{D}^{\geq 0}$ *as follows.*

$$\begin{aligned} \mathcal{D}^{\leq 0} &:= \{ X \in \mathcal{D} \mid F_L X \in \mathcal{A}^{\leq 0}, GX \in \mathcal{B}^{\leq 0} \}, \\ \mathcal{D}^{\geq 0} &:= \{ X \in \mathcal{D} \mid F_R X \in \mathcal{A}^{\geq 0}, GX \in \mathcal{B}^{\geq 0} \}. \end{aligned}$$

Then the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ *is a t−structure on* D.

Proof. We show that the conditions TS 1－TS 3 in Definition 2.1 of [Ter 15] are satisfied.
（TS 1）　We have

$$
\begin{aligned}
T\mathcal{D}^{\leq 0} &= \{ TX \mid X \in \mathcal{D}, F_L X \in \mathcal{A}^{\leq 0},\ GX \in \mathcal{B}^{\leq 0} \} \\
&= \{ Y \in \mathcal{D} \mid F_L T^{-1} Y \in \mathcal{A}^{\leq 0},\ GT^{-1} Y \in \mathcal{B}^{\leq 0} \} \\
&= \{ Y \in \mathcal{D} \mid T^{-1} F_L Y \in \mathcal{A}^{\leq 0},\ T^{-1} GY \in \mathcal{B}^{\leq 0} \} \\
&= \{ Y \in \mathcal{D} \mid F_L Y \in T\mathcal{A}^{\leq 0},\ GY \in T\mathcal{B}^{\leq 0} \} \\
&\subseteq \{ Y \in \mathcal{D} \mid F_L Y \in \mathcal{A}^{\leq 0},\ GY \in \mathcal{B}^{\leq 0} \} \\
&= \mathcal{D}^{\leq 0}
\end{aligned}
$$

and

$$
\begin{aligned}
T^{-1}\mathcal{D}^{\geq 0} &= \{ T^{-1}X \mid X \in \mathcal{D}, F_R X \in \mathcal{A}^{\geq 0},\ GX \in \mathcal{B}^{\geq 0} \} \\
&= \{ Y \in \mathcal{D} \mid F_R T Y \in \mathcal{A}^{\geq 0},\ GT Y \in \mathcal{B}^{\geq 0} \} \\
&= \{ Y \in \mathcal{D} \mid TF_R Y \in \mathcal{A}^{\geq 0},\ T GY \in \mathcal{B}^{\geq 0} \} \\
&= \{ Y \in \mathcal{D} \mid F_R Y \in T^{-1}\mathcal{A}^{\geq 0},\ GY \in T^{-1}\mathcal{B}^{\geq 0} \} \\
&\subseteq \{ Y \in \mathcal{D} \mid F_R Y \in \mathcal{A}^{\geq 0},\ GY \in \mathcal{B}^{\geq 0} \} \\
&= \mathcal{D}^{\geq 0}
\end{aligned}
$$

(TS 2）From the distinguished triangle (3.1),we obtain a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(FF_L X, Y) \to \mathrm{Hom}_{\mathcal{D}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(G_L GX, Y) \to \cdots.$$

If $X \in \mathcal{D}^{\leq 0}$, then $F_L X \in \mathcal{A}^{\leq 0}$ and $GX \in \mathcal{B}^{\leq 0}$. If $Y \in T^{-1}\mathcal{D}^{\geq 0}$, then $F_R T Y \in \mathcal{A}^{\geq 0}$ and $G T Y \in \mathcal{B}^{\geq 0}$, and thus $TF_R Y \in \ \in \mathcal{A}^{\geq 0}$ and $G T Y \in \mathcal{B}^{\leq 0}$. Hence we obtain $F_R Y \in T^{-1}\mathcal{A}^{\geq 0}$ and $G Y \in T^{-1}\mathcal{B}^{\geq 0}$. Then we have

$$\mathrm{Hom}_{\mathcal{D}}(FF_L X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(F_L X, F_R Y) = 0$$

and

$$\mathrm{Hom}_{\mathcal{D}}(G_L GX, Y) \simeq \mathrm{Hom}_{\mathcal{B}}(GX, GY) = 0.$$

Therefore $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for any $X \in \mathcal{D}^{\leq 0}$ and $Y \in T^{-1}\mathcal{D}^{\geq 0}$.
(TS 3) Let $X \in \mathcal{D}$. From Proposition 2.9. (*iii*) of [Ter 15], we have a distinguished triangle in $\mathcal{B}$

(3.3) $$\tau^{\mathcal{B}}_{\leq 0} GX \xrightarrow{\eta^0_{\mathcal{B}}(GX)} GX \xrightarrow{\varepsilon^1_{\mathcal{B}}(GX)} \tau^{\mathcal{B}}_{\geq 1} GX \xrightarrow{d^1_{\mathcal{B}}(GX)} T\tau^{\mathcal{B}}_{\leq 0} GX.$$

Hence we obtain a distinguished triangle

(3.4) $$Y \xrightarrow{g} X \xrightarrow{G_R(\varepsilon^1_{\mathcal{B}}(GX)) \circ \varepsilon_G(X)} G_R \tau^{\mathcal{B}}_{\geq 1} GX \to TY,$$

where $\varepsilon_G : \mathrm{id}_{\mathcal{D}} \to G_R \circ G$ is the counit of the adjunction.

Similarly, we have a distinguished triangle in $\mathcal{A}$

(3.5) $$\tau^{\mathcal{A}}_{\leq 0} F_L Y \xrightarrow{\eta^0_{\mathcal{A}}(F_L Y)} F_L Y \xrightarrow{\varepsilon^1_{\mathcal{A}}(F_L Y)} \tau^{\mathcal{A}}_{\geq 1} F_L Y \xrightarrow{d^1_{\mathcal{A}}(F_L Y)} T\tau^{\mathcal{A}}_{\leq 0} F_L Y.$$

Hence we obtain a distinguished triangle

(3.6) $$A \xrightarrow{f} Y \xrightarrow{F(\varepsilon^1_{\mathcal{A}}(F_L Y)) \circ \varepsilon_F(Y)} F\tau^{\mathcal{A}}_{\geq 1} F_L Y \to TA,$$

where $\varepsilon_F : \mathrm{id}_{\mathcal{D}} \to F \circ F_L$ is the counit of the adjunction.

Then we have the octahedron diagram for distinguished triangles (3.4) and (3.6)

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & Y & \longrightarrow & F\tau^{\mathcal{A}}_{\geq 1}F_L Y & \longrightarrow & TA \\
\| & & \downarrow{\scriptstyle g} & & \downarrow & & \| \\
A & \xrightarrow{g\circ f} & X & \longrightarrow & B & \longrightarrow & TA \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{g} & X & \longrightarrow & G_R\tau^{\mathcal{B}}_{\geq 1}GX & \longrightarrow & TY \\
\downarrow & & \downarrow & & \| & & \downarrow \\
F\tau^{\mathcal{A}}_{\geq 1}F_L Y & \longrightarrow & B & \longrightarrow & G_R\tau^{\mathcal{B}}_{\geq 1}GX & \longrightarrow & TF\tau^{\mathcal{A}}_{\geq 1}F_L Y .
\end{array}
$$

Applying F_L to the distinguished triangle (3.6), we have

$$
(3.7) \qquad F_L A \xrightarrow{F_L(f)} F_L Y \xrightarrow{F_L(F(\varepsilon^1_{\mathcal{A}}(F_L Y))\circ\varepsilon_F(Y))} F_L F\tau^{\mathcal{A}}_{\geq 1}F_L Y \longrightarrow TF_L\mathcal{A},
$$

From Proposition 2.4. (i) of [Ter 16], we see that $F_L(F(\varepsilon^1_{\mathcal{A}}(F_L Y))\circ\varepsilon_F(Y))$ factors as follows:

$$
\begin{array}{ccccc}
F_L Y & & \xrightarrow{F_L(F(\varepsilon^1_{\mathcal{A}}(F_L Y))\circ\varepsilon_F(Y))} & & F_L F\tau^{\mathcal{A}}_{\geq 1}F_L Y \\
\| & \searrow^{F_L(\varepsilon_F(Y))} & & & \| \\
\| & & F_L F F_L Y & \xrightarrow{F_L F(\varepsilon^1_{\mathcal{A}}(F_L Y))} & F_L F\tau^{\mathcal{A}}_{\geq 1}F_L Y \\
\| & \swarrow_{\eta_F(F_L Y)} & & & \simeq\downarrow{\scriptstyle \eta_F(\tau^{\mathcal{A}}_{\geq 1}F_L Y)} \\
F_L Y & & \xrightarrow[(\varepsilon^1_{\mathcal{A}}(F_L Y)]{} & & \tau^{\mathcal{A}}_{\geq 1}F_L Y .
\end{array}
$$

Hence the distinguished triangle (3.7) is isomorphic to the distinguished triangle (3.5). Therefore by Proposition 2.9. (iii) of [Ter 15], we obtain $F_L A \simeq \tau^{\mathcal{A}}_{\leq 0}F_L Y \in \mathcal{A}^{\leq 0}$, and thus $F_L A \in \mathcal{A}^{\geq 0}$.

Applying G to the distinguished triangle of the forth row in the above octahedron diagram, we have

$$
GF\tau^{\mathcal{A}}_{\geq 1}F_L Y \to GB \to GG_R\tau^{\mathcal{B}}_{\geq 1}GX \to TGF\tau^{\mathcal{A}}_{\geq 1}F_L Y .
$$

Since $G\circ F = 0$ and $\eta_G : G\circ G_R \xrightarrow{\sim} \mathrm{id}_{\mathcal{B}}$, we obtain $GB \simeq \tau^{\mathcal{B}}_{\geq 1}GX \in \mathcal{B}^{\geq 1} = T^{-1}\mathcal{B}^{\geq 0}$, and thus $GB \in T^{-1}\mathcal{B}^{\geq 0}$.

Applying G to the distinguished triangle (3.4), we have

$$
(3.8) \qquad GY \xrightarrow{G(g)} GX \xrightarrow{G(G_R(\varepsilon^1_{\mathcal{B}}(GX))\circ\varepsilon_G(X))} GG_R\tau^{\mathcal{B}}_{\geq 1}GX \to TGY .
$$

From Proposition 2.4. (i) of [Ter 16], we see that $G(G_R(\varepsilon^1_{\mathcal{B}}(GX))\circ\varepsilon_G(X))$ factors as follows:

$$
\begin{array}{ccccc}
GX & & \xrightarrow{G(G_R(\varepsilon^1_{\mathcal{B}}(GX))\circ\varepsilon_G(X))} & & GG\tau^{\mathcal{B}}_{\geq 1}GX \\
\| & \searrow^{G(\varepsilon_G(X))} & & & \| \\
\| & & GG_R GX & \xrightarrow{GG_R(\varepsilon^1_{\mathcal{B}}(GX))} & GG\tau^{\mathcal{B}}_{\geq 1}GX \\
\| & \swarrow_{\eta_F(GX)} & & & \simeq\downarrow{\scriptstyle \eta_G(\tau^{\mathcal{B}}_{\geq 1}GX)} \\
GX & & \xrightarrow[(\varepsilon^1_{\mathcal{B}}(GX)]{} & & \tau^{\mathcal{B}}_{\geq 1}GX .
\end{array}
$$

Hence the distinguished triangle (3.8) is isomorphic to the distinguished triangle (3.3). Therefore by Proposition 2.9. (iii) of [Ter 15], we obtain $GY \simeq \tau^{\mathcal{B}}_{\geq 0}GX \in \mathcal{B}^{\geq 0}$. Moreover, applying G to the distinguished triangle (3.6), it follows from $G\circ F = 0$ that $GA \simeq GY \in \mathcal{B}^{\geq 0}$, and thus $GA \in \mathcal{B}^{\geq 0}$.

Applying F_R to the distinguished triangle of the forth row in the above octahedron diagram, we have

$$F_R F \tau_{\geq 1}^{\mathcal{A}} F_L Y \to F_R B \to F_R G_R \tau_{\geq 1}^{\mathcal{B}} GX \to TF_R F \tau_{\geq 1}^{\mathcal{A}} F_L Y.$$

Since $F_R \circ G_R = 0$ and $\varepsilon'_F : \mathrm{id}_{\mathcal{A}} \xrightarrow{\sim} F_R \circ F$, we obtain $F_R B \simeq \tau_{\geq 1}^{\mathcal{A}} F_L Y \in \mathcal{A}^{\geq 1} = T^{-1}\mathcal{A}^{\geq 0}$, and thus $F_R B \in T^{-1}\mathcal{A}^{\geq 0}$.

Therefore, for any $X \in \mathcal{D}$, we have a distinguished triangle

$$A \to X \to B \to TA$$

with $A \in \mathcal{D}^{\leq 0}$ and $B \in T^{-1}\mathcal{D}^{\geq 0}$. □

Remark 3.3. Let $X \in \mathcal{D}$. Under the avove conditions GL 1－GL 5, we have the following:

$$\begin{aligned} F_L X \in \mathcal{A}^{\leq 0} &\Leftrightarrow F_L X \in {}^{\perp}(\mathcal{A}^{\geq 1})_{\mathcal{A}} \\ &\Leftrightarrow \mathrm{Hom}_{\mathcal{A}}(F_L X, Y) = 0 \quad \text{for all } Y \in \mathcal{A}^{\geq 1} \\ &\Leftrightarrow \mathrm{Hom}_{\mathcal{D}}(X, FY) = 0 \quad \text{for all } Y \in \mathcal{A}^{\geq 1} \end{aligned}$$

and

$$\begin{aligned} F_R X \in \mathcal{A}^{\geq 0} &\Leftrightarrow F_R X \in {}^{\perp}(\mathcal{A}^{\leq -1})_{\mathcal{A}} \\ &\Leftrightarrow \mathrm{Hom}_{\mathcal{A}}(Y, F_R X) = 0 \quad \text{for all } Y \in \mathcal{A}^{\leq -1} \\ &\Leftrightarrow \mathrm{Hom}_{\mathcal{D}}(FY, X) = 0 \quad \text{for all } Y \in \mathcal{A}^{\leq -1}. \end{aligned}$$

If F is the inclusion functor, then the above conditions are equivalent to the following conditions

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0 \quad \text{for all } Y \in \mathcal{A}^{>0}$$

and

$$\mathrm{Hom}_{\mathcal{D}}(Y, X) = 0 \quad \text{for all } Y \in \mathcal{A}^{<0},$$

respectively. □

4 Perverse t-structures

Let $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories and let $F : \mathcal{A} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{B}$ be exact functors. In particular, we assume that $\mathcal{A}$ is the kernel of G and F is the inclusion functor.

Lemma 4.1. *We have the following.*

(*i*) *If G admits a left adjoint functor G_L, then* $\mathrm{Ker}(G) = (G_L\mathcal{B})^{\perp}$.

(*ii*) *If G admits a right adjoint functor G_R, then* $\mathrm{Ker}(G) = {}^{\perp}(G_R\mathcal{B})$.

Proof. By Yoneda Lemma, we obtain

$$\begin{aligned} \mathrm{Ker}(G) &= \{X \in \mathcal{D} \mid GX \cong 0\} \\ &= \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{B}}(Y, GX) = 0 \text{ for all } Y \in \mathcal{B}\} \\ &= \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(G_L Y, X) = 0 \text{ for all } Y \in \mathcal{B}\} \\ &= (G_L\mathcal{B})^{\perp} \end{aligned}$$

if G admits a left adjoint functor G_L, and

$$\begin{aligned} \mathrm{Ker}(G) &= \{X \in \mathcal{D} \mid GX \cong 0\} \\ &= \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{B}}(GX, Y) = 0 \text{ for all } Y \in \mathcal{B}\} \\ &= \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(X, G_R Y) = 0 \text{ for all } Y \in \mathcal{B}\} \\ &= {}^{\perp}(G_R\mathcal{B}) \end{aligned}$$

if G admits a right adjoint functor G_R, respectively. □

Proposition 4.2. *We assume that the following conditions hold*:

(*i*) *G admits a left adjoint functor GL and a right adjoint functor G_R,*

(*ii*) *the functors G_L and G_R are both fully faithful.*

Then the above conditions GL1—GL5 hold.

Proof. We verify the above conditions GL1—GL5.

(GL1) Since G_L is fully faithful, we have the exact triple $G_L\mathcal{B} \xrightarrow{\iota_L} \mathcal{D} \xrightarrow{Q_L} \mathcal{D}/G_L\mathcal{B}$, where $G_L\mathcal{B}$ is an essential image of G_L. By Proposition 5.8 of [Ter 16] and Lemma 4.1, the functor $p_L := Q_L \circ F : \mathcal{A} \to \mathcal{D}/G_L\mathcal{B}$ is an exact equivalece
and the Verdier localization functor Q_L has a right adjoint functor $(Q_L)_* := F \circ p_L^{-1}$, where p_L^{-1} is the quasi-inverse functor to p_L. Put $F_L := p_L^{-1} \circ Q_L$. Then we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(F_L X, Y) &= \mathrm{Hom}_{\mathcal{A}}(p_L^{-1} Q_L X, Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}/G_L\mathcal{B}}(Q_L X, p_L Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X, (Q_L)_* p_L Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X, F p_L^{-1} p_L Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X, FY)\end{aligned}$$

for any $X \in \mathcal{D}$ and any $Y \in \mathcal{A}$. Hence F_L is a left adjoint functor to F.

Similarly, since G_R is fully faithful, we have the exact triple $G_R\mathcal{B} \xrightarrow{\iota_R} \mathcal{D} \xrightarrow{Q_R} \mathcal{D}/G_R\mathcal{B}$, where $G_R\mathcal{B}$ is an essential image of G_R. By Proposition 5.9 of [Ter 16] and Lemma 4.1, the functor $p_R := Q_R \circ F : \mathcal{A} \to \mathcal{D}/G_R\mathcal{B}$ is an exact equivalece and the Verdier localization functor Q_R has a right adjoint functor $(Q_R)^! := F \circ p_R^{-1}$, where p_R^{-1} is the quasi-inverse functor to p_R. Put $F_R := p_R^{-1} \circ Q_R$. Then we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(X, F_R Y) &= \mathrm{Hom}_{\mathcal{A}}(X, p_R^{-1} Q_R Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}/G_R\mathcal{B}}(p_R X, Q_R Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}}((Q_L)^! p_R X, Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(F p_R^{-1} p_R X, Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(FX, Y)\end{aligned}$$

for any $X \in \mathcal{A}$ and any $Y \in \mathcal{D}$. Hence F_R is a right adjoint functor to F.

(GL2) This follows from the assumption $G_L \dashv G \dashv G_R$.

(GL3) This follows from the construction of the functor F.

(GL4) Since G is a right adjoint functor to G_L, by Proposition 5.8 of [Ter 16], we have the distinguished triangle

$$X_{G_L B} \to X \to X_{(G_L B)^\perp} \to TX_{G_L B}.$$

By Lemma 4.1. (*i*), we have $(G_L B)^\perp = \mathrm{Ker}(G) \simeq F\mathcal{A}$. There exist an object $Y \in \mathcal{B}$ with $X_{G_L B} = G_L Y$ and an object $Z \in \mathcal{A}$ with $X_{(G_L B)^\perp} \simeq FZ$. Hence we obtain the distinguished triangle

$$G_L Y \to X \to FZ \to TG_L Y. \tag{4.1}$$

Then, applying G to (4.1), we have

$$GG_L Y \to GX \to GFZ \to TGG_L Y.$$

Since $GG_L Y \simeq Y$ by Proposition 2.8. (*i*) of [Ter 16] and $GFZ = 0$, we see that $Y \simeq GX$. Moreover, applying F_L to (4.1), we have

$$F_L G_L Y \to F_L X \to F_L F Z \to TF_L G_L Y.$$

Since $F_L G_L Y = 0$ by Proposition 3.1. (*i*) and $F_L FZ \simeq Z$ by Proposition 3.1. (*iii*), we see that $Z \simeq F_L X$. Therefore, by Proposition 3.1. (*ii*), we obtain the distinguished triangle

$$G_L GX \to X \to FF_L X \xrightarrow{d'} TG_L GX.$$

Similarly, since G is a left adjoint functor to G_R, by Proposition 5.9 of [Ter 16], we have the distinguished triangle

$$X_{\perp(G_R\mathcal{B})} \to X \to X_{G_R\mathcal{B}} \to T X_{\perp(G_R\mathcal{B})}.$$

By Lemma 4.1. (*ii*), we have ${}^{\perp}(G_L\mathcal{B}) = \mathrm{Ker}(G) \simeq F\mathcal{A}$. There exist an object $V \in \mathcal{A}$ with $FV = X_{\perp(G_R\mathcal{B})}$ and an object $W \in \mathcal{B}$ with $G_R W \simeq X_{G_R\mathcal{B}}$. Hence we obtain the distinguished triangle

(4.2) $$FV \to X \to G_R W \to TFV.$$

Then, applying G to (4.2), we have

$$GFV \to GX \to GG_R W \to TGFV.$$

Since $GFV = 0$ and $GG_R W \simeq W$ by Proposition 2.8. (*ii*) of [Ter 16], we see that $W \simeq GX$. Moreover, applying F_R to (4.2), we have

$$F_R F V \to F_R X \to F_R G_R W \to TF_R F V.$$

Since $F_R F V \simeq V$ by Proposition 3.1. (*iii*) and $F_R G_R W = 0$ by Proposition 3.1. (*i*), we see that $V \simeq F_R X$. Therefore, by Proposition 3.1. (*ii*), we obtain the distinguished triangle

$$FF_R X \to X \to G_R GX \xrightarrow{d'} TFF_R X.$$

(GL 5)　F is fully faithful because it is the inclusion functor. G_L and G_R are fully faithful from the assumption. □

Definition 4.3. Let $\mathcal{D}$ and $\mathcal{B}$ be triangulated categories with t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ and $(\mathcal{B}^{\leq 0}, \mathcal{B}^{\geq 0})$ on $\mathcal{B}$, respectively. Let $G : \mathcal{D} \to \mathcal{B}$ be a t-exact functor and let $\mathcal{A}$ be the kernel of G. Let $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0}) := (\mathcal{A} \cap \mathcal{D}^{\leq 0}, \mathcal{A} \cap \mathcal{D}^{\geq 0})$ be the induced t-structure on $\mathcal{A}$. Assume that G admits a left adjoint functor and a right adjoint functor, and the adjoint functors are both fully faithful. For an integer p, we obtain a new t-structure $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ on $\mathcal{D}$ by glueing t-structures $(\mathcal{A}^{\leq p}, \mathcal{A}^{\geq p})$ and $(\mathcal{B}^{\leq 0}, \mathcal{B}^{\geq 0})$ from Theorem 3.2 and Proposition 4.2, where

$${}^p\mathcal{D}^{\leq 0} := \{X \in D \mid GX \in \mathcal{B}^{\leq 0} \text{ and } \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0 \text{ for all } Y \in \mathcal{A}^{>p}\},$$
$${}^p\mathcal{D}^{\geq 0} := \{X \in D \mid GX \in \mathcal{B}^{\geq 0} \text{ and } \mathrm{Hom}_{\mathcal{D}}(Y, X) = 0 \text{ for all } Y \in \mathcal{A}^{<p}\}.$$

Then the t-structure $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is called the perverse t-structure with a perversity p obtained by glueing t-structures $(\mathcal{A}^{\leq p}, \mathcal{A}^{\geq p})$ and $(\mathcal{B}^{\leq 0}, \mathcal{B}^{\geq 0})$ and its heart is the abelian category

$${}^p\mathrm{Per}\,(\mathcal{D}/\mathcal{B}) := {}^p\mathcal{D}^{\leq 0} \cap {}^p\mathcal{D}^{\geq 0}.$$

In particular, when $p = -1$, we put

$$\mathrm{Per}\,(\mathcal{D}/\mathcal{B}) := {}^{-1}\mathrm{Per}\,(\mathcal{D}/\mathcal{B})$$

and objects of Per $(\mathcal{D}/\mathcal{B})$ are called perverse ob jects. □

The following example is from algebraic geometry. For a projective variety X over the complex numbers C, we denote by $D(X)$ the derived category of complexes of quasi-coherent $\mathcal{O}_X$-modules with coherent cohomology sheaves. The derived category $D(X)$ has the standard t-structure.

Example 4.4. ([Bri 02]) Let $f: Y \to X$ be a birational morphism of projective varieties such that $\mathbf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$. By Grothendieck-Verdier duality theorem, the functor $\mathbf{R}f_*: D(Y) \to D(X)$ has the left adjoint functor $\mathbf{L}f^*$ and the right adjoint functor $f^!$. Note that the functor $\mathbf{L}f^*$ is fully faithful, and then the functor $f^!$ is also fully faithful, because the dualizing functors $\mathbf{D}_X$ and $\mathbf{D}_Y$ are autoequivalences: $f^! = \mathbf{D}_Y \circ \mathbf{L}f^* \circ \mathbf{D}_X$.

Now let us write $\mathcal{D} = D(Y)$, $\mathcal{B} = D(X)$ and $G = \mathbf{R}f_*$, and then we put $\mathcal{A} = \mathrm{Ker}(G)$. Then we are in the situation of Definition 4.3. A complex $F^\bullet \in D(Y)$ lies in Ker $(\mathbf{R}f_*)$ if and only if its cohomology sheaves $H^i(F^\bullet)$ lies in Ker $(\mathbf{R}f_*)$. For a complex

$$F^\bullet : \cdots \to F^{-2} \to F^{-1} \to F^0 \xrightarrow{d^0} F^1 \to \cdots$$

of Ker $(\mathbf{R}f_*)$, we see that the truncation complex

$$\tau^{D(Y)}_{\leq 0}(F^\bullet) : \cdots \to F^{-2} \to F^{-1} \to Ker(d^0) \to 0 \to \cdots$$

belongs to Ker $(\mathbf{R}f_*)$. We have the induced t-structure (Ker $(\mathbf{R}f_*)^{\leq 0}$, Ker $(\mathbf{R}f_*)^{\geq 0}$) on Ker $(\mathbf{R}f_*)$ from the standard t-structure $(D(Y)^{\leq 0}, D(Y)^{\geq 0})$ on $D(Y)$ by Proposition 2.5. Hence we ob-tain the perverse t-structure $({}^pD(Y)^{\leq 0}, {}^pD(Y)^{\geq 0})$ on $D(Y)$ with a perversity p by glueing the induced t-structure (Ker $(\mathbf{R}f_*)^{\leq p}$, Ker$(\mathbf{R}f_*)^{\geq p}$) on Ker $(\mathbf{R}f_*)$ and the standard t-structure $(D(X)^{\leq 0}, D(X)^{\geq 0})$ on $D(X)$. Then objects of Per (Y/X) := Per $(D(Y)/D(X))$ are called per-verse coherent sheaves. □

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