

<研究ノート>

Triangulated Categories IV:

Admissible Subcategories

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要旨

この概説は、主に代数幾何学に現れる三角圏や導来圏の性質についてまとめた一連の研究ノートの第4部であり、今回は三角圏の許容部分圏に関して解説する。許容部分圏はモビリティの貼り合わせや代数多様体上の（代数的）偏屈層を定義するために必要となる重要な概念である。

第2節において随伴関手の定義と基本性質を、第3節と第4節では研究ノートの第3部に対する補足として加法圏と三角圏の局所化を解説する。最後に第5節では許容部分圏の定義を与え、その性質と三角圏の半直交分解との関係について解説する。

1 Introduction

This exposition is the fourth part of our study of triangulated categories in algebraic geometry. We shall explain the definitions and fundamental properties of adjoint functors and admissible subcategories of triangulated categories, and describe their proofs in detail.

2 Adjoint Functors

Definition 2.1. Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ be functors. We say that the pair (L, R) is a pair of **adjoint functors**, or L is a **left adjoint** functor to R , or R is a **right adjoint** functor to L , if there exists an isomorphism Φ of bifunctors from $\mathcal{C}^{\text{op}} \times \mathcal{D}$ to $\mathcal{S}et$:

$$(2.1) \quad \text{Hom}_{\mathcal{D}}(L(\bullet), \bullet) \simeq \text{Hom}_{\mathcal{C}}(\bullet, R(\bullet)).$$

We call the above isomorphism the **adjunction isomorphism**. If (L, R) is a pair of adjoint functors, we often denote it by $L \dashv R$. □

Theorem 2.2. Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ be functors. Then we have the following.

- (i) If L admits a right adjoint functor, this adjoint functor is unique up to unique isomorphism.
- (ii) If R admits a left adjoint functor, this adjoint functor is unique up to unique isomorphism.

Moreover, a functor L admits a right adjoint functor if and only if the functor $\text{Hom}_{\mathcal{D}}(L(\bullet), Y)$ is representable for any $Y \in \mathcal{D}$.

Proof. This follows from the Yoneda lemma. \square

Let X be any object of \mathcal{C} . Applying the isomorphism (2.1) with X and $L(X)$, we find the isomorphism

$$\Phi_{X, L(X)} : \text{Hom}_{\mathcal{D}}(L(X), L(X)) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X, RL(X))$$

and the identity morphism $\text{id}_{L(X)}$ of $L(X)$ defines a morphism $X \Rightarrow RL(X)$. Similarly, we have the isomorphism

$$\Phi_{R(X), Y}^{-1} : \text{Hom}_{\mathcal{C}}(R(Y), R(Y)) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(LR(Y), Y)$$

and the identity morphism $\text{id}_{R(Y)}$ of $R(Y)$ defines a morphism $LR(Y) \Rightarrow Y$. These morphisms are functorial with respect to X and Y . Hence we obtain constructed morphisms of functors

$$\varepsilon : \text{id}_{\mathcal{C}} \Rightarrow R \circ L \quad \text{and} \quad \eta : L \circ R \Rightarrow \text{id}_{\mathcal{D}}.$$

The morphism ε (resp. η) is called the **counit** (resp. **unit**) of the adjunction Φ .

Proposition 2.3. Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \Rightarrow \mathcal{D}$ and $R : \mathcal{D} \Rightarrow \mathcal{C}$ be functors. Assume that L is a left adjoint functor to R . Then the adjunction isomorphism

$$\Phi_{X, Y} : \text{Hom}_{\mathcal{D}}(L(X), Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X, R(Y))$$

for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ can be described as

$$\Phi_{X, Y}(f) = R(f) \circ \varepsilon_X \quad \text{and} \quad \Phi_{X, Y}^{-1}(g) = \eta_Y \circ L(g)$$

for $f : L(X) \Rightarrow Y$ and $g : X \Rightarrow R(Y)$.

Proof. Since the adjunction isomorphism Φ is an isomorphism of bifunctors, we have the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(L(X), L(X)) & \xrightarrow{\Phi_{X, L(X)}} & \text{Hom}_{\mathcal{C}}(X, RL(X)) \\ f_* \downarrow & & \downarrow R(f)_* \\ \text{Hom}_{\mathcal{D}}(L(X), Y) & \xrightarrow{\Phi_{X, Y}} & \text{Hom}_{\mathcal{C}}(X, R(Y)) \end{array}$$

for $f : L(X) \Rightarrow Y$. Then we obtain

$$\begin{aligned} \Phi_{X, Y}(f) &= (\Phi_{X, Y} \circ f_*)(\text{id}_{L(X)}) \\ &= (R(f)_* \circ \Phi_{X, L(X)})(\text{id}_{L(X)}) \\ &= R(f)_*(\Phi_{X, L(X)}(\text{id}_{L(X)})) \\ &= R(f)_*(\varepsilon_X) \\ &= R(f) \circ \varepsilon_X. \end{aligned}$$

Similarly, we have the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(LR(Y), Y) & \xrightarrow{\Phi_{R(X), Y}} & \text{Hom}_{\mathcal{C}}(R(Y), R(Y)) \\ L(g)_* \downarrow & & \downarrow g^* \\ \text{Hom}_{\mathcal{D}}(L(X), Y) & \xrightarrow{\Phi_{X, Y}} & \text{Hom}_{\mathcal{C}}(X, R(Y)) \end{array}$$

for $g : X \rightarrow R(Y)$. Then we obtain

$$\begin{aligned}
 \Phi_{X,Y}^{-1}(g) &= (\Phi_{X,Y}^{-1} \circ g^*) (\text{id}_R(Y)) \\
 &= (L(g)^* \circ \Phi_{R(X),Y}^{-1}) (\text{id}_R(Y)) \\
 &= L(g)^* (\Phi_{R(X),Y}^{-1} (\text{id}_{R(Y)})) \\
 &= L(g)^* (\eta_Y) \\
 &= \eta_Y \circ L(g).
 \end{aligned}$$

This completes the proof.

Proposition 2.4. *Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ be functors. Assume that L is a left adjoint functor to R . If $\varepsilon : \text{id}_{\mathcal{C}} \rightarrow R \circ L$ is the counit and $\eta : L \circ R \rightarrow \text{id}_{\mathcal{D}}$ is the unit of the adjunction, then we have the following.*

(i) *The composition*

$$L \xrightarrow{L \circ \varepsilon} L \circ (R \circ L) = (L \circ R) \circ L \xrightarrow{\eta \circ L} L$$

is the identity id_L .

(ii) *The composition*

$$R \xrightarrow{\varepsilon \circ R} (R \circ L) \circ R = R \circ (L \circ R) \xrightarrow{R \circ \eta} R$$

is the identity id_R .

Proof. We give only the proof of (i), since the proof of (ii) is similar to that of (i).

By Proposition 2.3, we have

$$\text{id}_{L(X)} = \Phi_{X,L(X)}^{-1}(\varepsilon_X) = \eta_{L(X)} \circ L(\varepsilon_X)$$

for any $X \in \mathcal{C}$. Hence we obtain

$$(\eta \circ L) \circ (L \circ \varepsilon) = \text{id}_L.$$

This completes the proof.

Proposition 2.5. *Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ be functors. Then we have the following commutative diagrams for $X, X' \in \mathcal{C}$ and $Y, Y' \in \mathcal{D}$.*

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{C}}(X, X') & \xrightarrow{L} & \text{Hom}_{\mathcal{D}}(L(X), L(X')) \\
 & \searrow (\varepsilon_{X'})^* & \downarrow \simeq \Phi_{X,L(X')} \\
 & & \text{Hom}_{\mathcal{C}}(X, RL(X'))
 \end{array}$$

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{D}}(Y, Y') & \xrightarrow{R} & \text{Hom}_{\mathcal{C}}(R(Y), R(Y')) \\
 & \searrow (\eta_Y)^* & \uparrow \simeq \Phi_{R(Y), Y'} \\
 & & \text{Hom}_{\mathcal{D}}(LR(Y), Y')
 \end{array}$$

Proof. We only show that the diagram (2.2) is commutative, since the proof of the commutativity of (2.3) is similar to that of (2.2).

Since the adjunction isomorphism Φ is an isomorphism of bifunctors, we have the

following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(L(X'), L(X')) & \xrightarrow{\Phi_{X', L(X')}} & \mathrm{Hom}_{\mathcal{C}}(X', RL(X')) \\ L(f)^* \downarrow & & \downarrow f^* \\ \mathrm{Hom}_{\mathcal{D}}(L(X), L(X')) & \xrightarrow{\Phi_{X, L(X')}} & \mathrm{Hom}_{\mathcal{C}}(X, RL(X')) \end{array}$$

for $f : X \rightarrow X'$. Then we obtain

$$\begin{aligned} (\Phi_{X, L(X')} \circ L)(f) &= \Phi_{X, L(X')}(L(f)) \\ &= \Phi_{X, L(X')}(L(f) * (\mathrm{id}_{L(X')})) \\ &= f^*(\Phi_{X' L(X')}(L(f))) \\ &= f^*(\varepsilon_{X'}) \\ &= \varepsilon_{X'} \circ f \\ &= (\varepsilon_{X'})_*(f). \end{aligned}$$

This completes the proof. \square

Proposition 2.6. Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ be functors. Let $\tilde{\varepsilon} : \mathrm{id}_{\mathcal{C}} \rightarrow R \circ L$ and $\tilde{\eta} : L \circ R \rightarrow \mathrm{id}_{\mathcal{D}}$ be morphisms of functors. Assume that $(\tilde{\eta} \circ L) \circ (L \circ \tilde{\varepsilon}) = \mathrm{id}_L$ and $(R \circ \tilde{\eta}) \circ (\tilde{\varepsilon} \circ R) = \mathrm{id}_R$. Then (L, R) is a pair of adjoint functors. Moreover $\tilde{\varepsilon}$ and $\tilde{\eta}$ are the counit and the unit of the adjunction, respectively.

Proof. For any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the two composite morphisms

$$\mathrm{Hom}_{\mathcal{D}}(L(X), Y) \xrightarrow{R} \mathrm{Hom}_{\mathcal{C}}(RL(X), R(Y)) \xrightarrow{(\tilde{\varepsilon}_X)^*} \mathrm{Hom}_{\mathcal{C}}(X, R(Y))$$

and

$$\mathrm{Hom}_{\mathcal{C}}(X, R(Y)) \xrightarrow{L} \mathrm{Hom}_{\mathcal{D}}(L(X), LR(Y)) \xrightarrow{(\tilde{\eta}_Y)^*} \mathrm{Hom}_{\mathcal{D}}(L(X), Y)$$

are inverse to each other. Indeed, since $\tilde{\varepsilon}$ and $\tilde{\eta}$ are morphisms of functors, we have the following commutative diagrams

$$\begin{array}{ccc} LRL(X) & \xrightarrow{\tilde{\eta}_{L(X)}} & L(X) \\ LR(f) \downarrow & & \downarrow \mathrm{id}_{\mathcal{D}}(f) = f \\ LR(Y) & \xrightarrow{\tilde{\eta}_Y} & Y, \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\tilde{\varepsilon}_X} & RL(X) \\ g = \mathrm{id}_{\mathcal{C}}(g) \downarrow & & \downarrow RL(g) \\ R(Y) & \xrightarrow{\tilde{\varepsilon}_{R(Y)}} & RLR(Y). \end{array}$$

Put $\Phi_{X, Y} := (\tilde{\varepsilon}_X)^* \circ R$ and $\Psi_{X, Y} := (\tilde{\eta}_Y)^* \circ L$. Then we obtain

$$\begin{aligned} (\Psi_{X, Y} \circ \Phi_{X, Y})(f) &= ((\tilde{\eta}_Y)^* \circ L \circ (\tilde{\varepsilon}_X)^* \circ R)(f) \\ &= ((\tilde{\eta}_Y)^* \circ L)(R(f) \circ \tilde{\varepsilon}_X) \\ &= (\tilde{\eta}_Y)^*(LR(f) \circ L(\tilde{\varepsilon}_X)) \\ &= (\tilde{\eta}_Y) LR(f) \circ L(\tilde{\varepsilon}_X) \\ &= f \circ \tilde{\eta}_{L(X)} \circ L(\tilde{\varepsilon}_X) \\ &= f \end{aligned}$$

and

$$\begin{aligned} (\Phi_{X, Y} \circ \Psi_{X, Y})(g) &= ((\tilde{\varepsilon}_X)^* \circ R \circ (\tilde{\eta}_Y)^* \circ L)(g) \\ &= ((\tilde{\varepsilon}_X)^* \circ R)(\tilde{\eta}_Y \circ L(g)) \end{aligned}$$

$$\begin{aligned}
 &= (\tilde{\varepsilon}_X)^* (R(\tilde{\eta}_Y) \circ RL(g)) \\
 &= R(\tilde{\eta}_Y) \circ RL(g) \circ \tilde{\varepsilon}_X \\
 &= R(\tilde{\eta}_Y) \circ \tilde{\varepsilon}_{R(Y)} \circ g \\
 &= g
 \end{aligned}$$

for any $f : L(X) \rightarrow Y$ and any $g : X \rightarrow R(Y)$. Hence $\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}$ and $\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}$. Thus (L, R) is a pair of adjoint functors and Φ is the adjunction isomorphism.

Moreover, if ε and η are the counit and the unit of the adjunction isomorphism Φ , respectively, then we obtain

$$\varepsilon_X = \Phi_{X,L(X)}(\text{id}_{L(X)}) = ((\tilde{\varepsilon}_X)^* \circ R)(\text{id}_{L(X)}) = \text{id}_{RL(X)} \circ \tilde{\varepsilon}_X = \tilde{\varepsilon}_X$$

and

$$\eta_Y = \Psi_{R(Y),Y}(\text{id}_{R(Y)}) = ((\tilde{\eta}_Y) \circ L)(\text{id}_{R(Y)}) = \tilde{\eta}_Y \circ \text{id}_{LR(Y)} = \tilde{\eta}_Y.$$

This completes the proof.

In the situation of Proposition 2.6, we say that $\langle L, R, \eta, \varepsilon \rangle$ is an **adjunction** and that the counit ε and the unit η are the **adjunction morphisms**.

Proposition 2.7. *Let \mathcal{C} , \mathcal{D} and \mathcal{E} be categories and let*

$$L : \mathcal{C} \rightarrow \mathcal{D}, \quad R : \mathcal{D} \rightarrow \mathcal{C}, \quad L' : \mathcal{D} \rightarrow \mathcal{E}, \quad R' : \mathcal{E} \rightarrow \mathcal{D}$$

be functors. If (L, R) and (L', R') are pairs of adjoint functors, then $(L' \circ L, R \circ R')$ is a pair of adjoint functors.

Proof. For any $X \in \mathcal{C}$ and any $Z \in \mathcal{E}$, we have functorial isomorphisms

$$\text{Hom}_{\mathcal{E}}(L' \circ L(X), Z) \simeq \text{Hom}_{\mathcal{D}}(L(X), R'(Z)) \simeq \text{Hom}_{\mathcal{C}}(X, R \circ R'(Z)).$$

This completes the proof. □

Proposition 2.8. *Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ be functors. If $\langle L, R, \eta, \varepsilon \rangle$ is an adjunction, then we have the following.*

- (i) *L is fully faithful if and only if $\varepsilon : \text{id}_{\mathcal{C}} \rightarrow R \circ L$ is an isomorphism.*
- (ii) *R is fully faithful if and only if $\eta : L \circ R \rightarrow \text{id}_{\mathcal{D}}$ is an isomorphism.*

Proof. We give only the proof of (i), since the proof of (ii) is similar to that of (i).

It follows from the diagram (2.2) that the following two conditions are equivalent:

- (1) the map $\text{Hom}_{\mathcal{C}}(X, X') \xrightarrow{L} \text{Hom}_{\mathcal{D}}(L(X), L(X'))$ is an isomorphism for any $X, X' \in \mathcal{C}$,
- (2) the map $\text{Hom}_{\mathcal{C}}(X, X') \xrightarrow{(\varepsilon_X)^*} \text{Hom}_{\mathcal{D}}(X, RL(X'))$ is an isomorphism for any $X, X' \in \mathcal{C}$.

Hence by the Yoneda lemma, we find that L is fully faithful if and only if ε_X is an isomorphism for all $X \in \mathcal{C}$.

Proposition 2.9. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an equivalence of categories. If $G : \mathcal{D} \rightarrow \mathcal{C}$ is a quasi-inverse functor to F , then G is a left and right adjoint functor to F .*

Proof. From the isomorphism $G \circ F \simeq \text{id}_C$, we have the following commutative diagram

$$\begin{array}{ccccc} \text{Hom}_{\mathcal{D}}(F(X'), Y) & \xrightarrow{G} & \text{Hom}_C(GF(X'), G(Y)) & \xrightarrow{\sim} & \text{Hom}_C(X', G(Y)) \\ F(f)^* \circ g_* \downarrow & & \downarrow GF(f)^* \circ G(g)_* & & \downarrow f^* \circ G(g)_* \\ \text{Hom}_{\mathcal{D}}(F(X), Y') & \xrightarrow{G} & \text{Hom}_C(GF(X), G(Y')) & \xrightarrow{\sim} & \text{Hom}_C(X, G(Y')) \end{array}$$

for $f : X \rightarrow X' \in \text{Mor}(C)$ and $g : Y \rightarrow Y' \in \text{Mor}(\mathcal{D})$. Hence we obtain an isomorphism of bifunctors

$$\text{Hom}_{\mathcal{D}}(F(\bullet), \bullet) \xrightarrow{\sim} \text{Hom}_C(\bullet, G(\bullet))$$

and thus $F \dashv G$.

Similarly, from the isomorphism $F \circ G \simeq \text{id}_{\mathcal{D}}$, we obtain $G \dashv F$. □

Proposition 2.10. *Let \mathcal{C} and \mathcal{D} be categories and let $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ be functors. If $\langle L, R, \eta, \varepsilon \rangle$ is an adjunction, then the following conditions are equivalent.*

- (i) *L is an equivalence of categories.*
- (ii) *R is an equivalence of categories.*
- (iii) *L and R are fully faithful.*

Proof. This follows from Proposition 2.8 and Proposition 2.9. □

3 Localization of additive categories

Proposition 3.1. *Let \mathcal{C} be a category and let \mathcal{S} be a multiplicative system in \mathcal{C} . If \mathcal{C} has a zero object 0 , then $\mathcal{C}_{\mathcal{S}}$ has also a zero object 0 .*

Proof. By definition, a zero object 0 is both initial and terminal, and thus for all $Z \in \mathcal{C}$, $\text{Hom}_{\mathcal{C}}(0, Z) \simeq \{\eta_Z\}$ and $\text{Hom}_{\mathcal{C}}(Z, 0) \simeq \{\xi_Z\}$.

(i) Let $f \in \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(0, 0)$. If $\xi_Y \in \mathcal{S}$ and $f = [(Y, \xi_Y, \xi_Y)]$, then we get $f = [(0, \text{id}_0, \text{id}_0)] = Q(\text{id}_0)$. This is visualized by the diagram

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow \xi_Y & \downarrow \xi_Y & \searrow \xi_Y & \\ 0 & \xleftarrow{\text{id}_0} & 0 & \xrightarrow{\text{id}_0} & 0 \end{array}$$

If there exists no $Y \in \mathcal{C}$ such that $Y \neq 0$ and $\xi_Y \in \mathcal{S}$, then we get $f = [(0, \text{id}_0, \text{id}_0)] = Q(\text{id}_0)$. Hence, in both cases, we obtain $\text{Hom}_{\mathcal{C}_{\mathcal{S}}}(0, 0) = \{Q(\text{id}_0)\}$.

(ii) If $X \in \mathcal{C}_{\mathcal{S}}$ and $f \in \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(0, X)$, then we get $f = [(X', \xi_{X'}, g)] = [(0, \text{id}_0, \eta_X)] = Q(\eta_X)$ in the same way. Hence we obtain $\text{Hom}_{\mathcal{C}_{\mathcal{S}}}(0, X) = \{Q(\eta_X)\}$. This is visualized by the diagram

$$\begin{array}{ccccc} & & X' & & \\ & \swarrow \xi_{X'} & \uparrow \eta_{X'} & \searrow g & \\ 0 & \xleftarrow{\text{id}_0} & 0 & \xrightarrow{\eta_X} & X \end{array}$$

(iii) If $X \in \mathcal{C}_{\mathcal{S}}$ and $f \in \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, 0)$, then we get $f = [(X', s, \xi_{X'})] = [(X, \text{id}_X, \xi_X)] = Q(\xi_X)$. Hence we obtain $\text{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, 0) = \{Q(\xi_X)\}$. This is visualized by the diagram

$$\begin{array}{ccccc}
 & & X' & & \\
 & \swarrow s & \downarrow s & \searrow \xi_{X'} & \\
 X & \xleftarrow{\text{id}_X} & X & \xrightarrow{\xi_X} & 0.
 \end{array}$$

Therefore 0 is a zero object in \mathcal{C}_S . □

Proposition 3.2. *Let \mathcal{C} be a category with a zero object 0 and let \mathcal{S} be a left multiplicative system in \mathcal{C} . Then for each object $X \in \mathcal{C}_S$, the followings are equivalent.*

- (i) $Q(X) \simeq 0$ in \mathcal{C}_S .
- (ii) *There exists an object $Y \in \mathcal{C}$ such that $0 : Y \rightarrow X$ belongs to \mathcal{S} .*

Proof. (i) \Rightarrow (ii). By Proposition 3.1, we see that $Q(\eta_X) = [(0, \text{id}_0, \eta_X)]$ is an isomorphism in \mathcal{C}_S , since $\text{Hom}_{\mathcal{C}_S}(0, X) = \{Q(\eta_X)\}$. Hence, by Lemma 4.5. (iv) of [Ter 14], there exists $Y \in \mathcal{C}$ such that $0 = \eta_X \circ \xi_Y : Y \rightarrow X \in \mathcal{S}$.

(ii) \Rightarrow (i). Since $0 = \eta_X \circ \xi_Y : Y \rightarrow X$ belongs to \mathcal{S} , we get $Q(0) = Q(\eta_X) \circ Q(\xi_Y) = 0$ is an isomorphism. Then we obtain $\text{id}_{Q(X)} = Q(0) \circ Q(0)^{-1} = 0 \circ Q(0)^{-1} = 0$. Hence $Q(X) \simeq 0$. □

Proposition 3.3. *Let \mathcal{C} be an additive category and let \mathcal{S} be a left multiplicative system in \mathcal{C} . Then the localization \mathcal{C}_S is an additive category and $Q : \mathcal{C} \rightarrow \mathcal{C}_S$ is an additive functor.*

Proof. At the beginning, let us check the axioms AD1–AD3 in Definition 2.12 of [OT14].

(AD1) By Proposition 3.1, there exists a zero object 0 in \mathcal{C}_S .

(AD2) First we show that for all $X, Y \in \mathcal{C}_S$, $\text{Hom}_{\mathcal{C}_S}(X, Y)$ has a structure of an additive group.

(i) Addition map: For each pair (X, Y) of objects $X, Y \in \mathcal{C}_S$, we define the addition map

$$+ : \text{Hom}_{\mathcal{C}_S}(X, Y) \times \text{Hom}_{\mathcal{C}_S}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}_S}(X, Y)$$

as follows: Let $f_1, f_2 \in \text{Hom}_{\mathcal{C}_S}(X, Y)$ be any morphisms. Assume that $f_1 = [(X_1, s_1, a_1)]$ and $f_2 = [(X_2, s_2, a_2)]$. By LMS3, we get a commutative diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{t_2} & X_2 \\
 t_1 \downarrow & & \downarrow s_2 \\
 X_1 & \xrightarrow{s_1} & X.
 \end{array}$$

in \mathcal{C} with $s_1 \circ t_1 = s_2 \circ t_2 \in \mathcal{S}$. Then we obtain

$f_1 = [(X_1, s_1, a_1)] = [(Y, s_1 \circ t_1, a_1 \circ t_1)]$ and $f_2 = [(X_2, s_2, a_2)] = [(Y, s_2 \circ t_2, a_2 \circ t_2)]$, and we define

$$f_1 + f_2 := [(Y, s_1 \circ t_1, a_1 \circ t_1 + a_2 \circ t_2)].$$

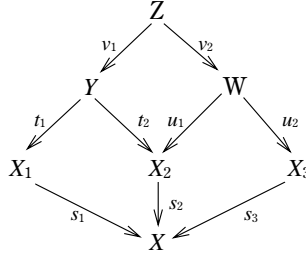
This is visualized by the diagram

$$\begin{array}{ccc}
 & Y & \\
 s_1 \circ t_1 \swarrow & & \searrow a_1 \circ t_1 + a_2 \circ t_2 \\
 X & & Y.
 \end{array}$$

(ii) Commutativity: We get

$$f_1 + f_2 = [(Y, s_1 \circ t_1, a_1 \circ t_1 + a_2 \circ t_2)] = [(Y, s_1 \circ t_1, s_2 \circ t_2 + a_1 \circ t_1)] = f_2 + f_1.$$

(iii) Associativity: Let $f_3 = [(X_3, s_3, a_3)] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$ be another morphism. By LMS3, we get a commutative diagram



in \mathcal{C} with $s_1 \circ t_1 \circ v_1 = s_2 \circ t_2 \circ v_1 = s_2 \circ u_1 \circ v_2 = s_3 \circ u_2 \circ v_2 \in \mathcal{S}$. Then we have

$$f_1 + f_2 = [(Y, s_1 \circ t_1, a_1 \circ t_1 + a_2 \circ t_2)] = [(Z, s_1 \circ t_1 \circ v_1, (a_1 \circ t_1 + a_2 \circ t_2) \circ v_1)]$$

and

$$f_3 = [(X_3, s_3, a_3)] = [(W, s_3 \circ u_2, a_3 \circ u_2)] = [(Z, s_3 \circ u_2 \circ v_2, a_3 \circ u_2 \circ v_2)].$$

Hence we get

$$(f_1 + f_2) + f_3 = [(Z, s_1 \circ t_1 \circ v_1, a_1 \circ t_1 \circ v_1 + a_2 \circ t_2 \circ v_1 + a_3 \circ u_2 \circ v_2)].$$

On the other hand, we have

$$f_2 + f_3 := [(W, s_2 \circ u_1, a_2 \circ u_1 + a_3 \circ u_2)] = [(Z, s_2 \circ u_1 \circ v_2, (a_2 \circ u_1 + a_3 \circ u_2) \circ v_2)]$$

and

$$f_1 = [(X_1, s_1, a_1)] = [(Y, s_1 \circ t_1, a_1 \circ t_1)] = [(Z, s_1 \circ t_1 \circ v_1, a_1 \circ t_1 \circ v_1)].$$

Hence we get

$$f_1 + (f_2 + f_3) = [(Z, s_1 \circ t_1 \circ v_1, a_1 \circ t_1 \circ v_1 + a_2 \circ u_1 \circ v_2 + a_3 \circ u_2 \circ v_2)].$$

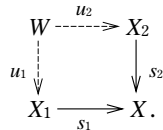
Since $t_2 \circ v_1 = u_1 \circ v_2$, we obtain $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(iv) Identity element: Let $f = [(X', s, f')] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$. Since $0 = [(X, \text{id}_X, 0)] = [(X', s, 0)]$, we get

$$f + 0 = [(X', s, f')] + [(X', s, 0)] = [(X', s, f' \circ \text{id}_{X'} + 0 \circ s)] = [(X', s, f')] = f.$$

(v) Inverse element: For $f = [(X', s, f')] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$, we put $-f := [(X', s, -f')]$. Then we obtain $f + (-f) = (-f) + f = 0$.

Next we show that the composition map \circ is bi-additive. Let $f_1 = [(X_1, s_1, a_1)]$, $f_2 = [(X_2, s_2, a_2)] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$ and $g = [(Y', t, b)] \in \text{Hom}_{\mathcal{C}_S}(Y, Z)$. Assume that $f_1 = [(X_1, s_1, a_1)]$ and $f_2 = [(X_2, s_2, a_2)]$. By LMS3, we get a commutative diagram



in \mathcal{C} with $s_1 \circ u_1 = s_2 \circ u_2 \in \mathcal{S}$. Then $f_1 + f_2 = [(Y, s_1 \circ u_1, a_1 \circ u_1 + a_2 \circ u_2)]$.

(AD3) For any $X, Y \in \mathcal{C}$, there exists the direct sum $X \oplus Y \in \mathcal{C}$ and morphisms

$$i_1 : X \rightarrow X \oplus Y, \quad i_2 : Y \rightarrow X \oplus Y, \quad p_1 : X \oplus Y \rightarrow X, \quad p_2 : X \oplus Y \rightarrow Y$$

such that

$$p_2 \circ i_1 = 0, \quad p_1 \circ i_2 = 0, \quad p_1 \circ i_1 = \text{id}_X, \quad p_2 \circ i_2 = \text{id}_Y$$

and

$$i_1 \circ p_1 + i_2 \circ p_2 = \text{id}_{X \oplus Y}.$$

Then for any $X, Y \in \text{Ob}(\mathcal{C}_S) = \text{Ob}(\mathcal{C})$, we have an object $Q(X \oplus Y) \in \mathcal{C}_S$ and morphisms

$$\begin{aligned} Q(i_1) : X &\rightarrow Q(X \oplus Y), \quad Q(i_2) : Y \rightarrow Q(X \oplus Y), \\ Q(p_1) : Q(X \oplus Y) &\rightarrow X, \quad Q(p_2) : Q(X \oplus Y) \rightarrow Y \end{aligned}$$

such that

$$Q(p_2) \circ Q(i_1) = 0, \quad Q(p_1) \circ Q(i_2) = 0, \quad Q(p_1) \circ Q(i_1) = \text{id}_X, \quad Q(p_2) \circ Q(i_2) = \text{id}_Y$$

and

$$Q(i_1) \circ Q(p_1) + Q(i_2) \circ Q(p_2) = \text{id}_{Q(X \oplus Y)}.$$

This means that $Q(X \oplus Y)$ is the direct sum $X \oplus Y$ of objects X and Y of \mathcal{C}_S .

Finally, we show that the functor Q is additive. This follows from the proof of (AD3) above.

For any $X, Y \in \text{Ob}(\mathcal{C}_S) = \text{Ob}(\mathcal{C})$, we obtain $Q(X \oplus Y) = Q(X) \oplus Q(Y)$. \square

4 Localization of triangulated categories

Let (\mathcal{D}, T) be a triangulated category and let \mathcal{S} be a multiplicative system in \mathcal{D} .

From Proposition 3.3, we have the following proposition.

Proposition 4.1. \mathcal{D}_S is an additive category and $Q : \mathcal{D} \rightarrow \mathcal{D}_S$ is an additive functor. \square

We define the automorphism T_S of \mathcal{D}_S as follows:

$$T_S(X) := T(X)$$

for any $X \in \text{Ob}(\mathcal{D}_S) = \text{Ob}(\mathcal{D})$, and

$$T_S([(X', s, f)]) := [(T(X'), T(s), T(f))]$$

for each morphism $[(X', s, f)] \in \text{Hom}_{\mathcal{D}_S}(X, Y)$, represented by the roof

$$\begin{array}{ccc} & X' & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

with $f \in \text{Hom}_{\mathcal{D}}(X', Y)$ and $s \in \mathcal{S}$.

Lemma 4.2. $T_S \circ Q = Q \circ T$.

Proof. For any $X \in \text{Ob}(\mathcal{D})$, we have

$$T_S(Q(X)) = T_S(X) = T(X) = Q(T(X)).$$

For any $f : X \rightarrow Y$ in \mathcal{D} , we have

$$T_S(Q(f)) = T_S([(X, \text{id}_X, f)]) = [(T(X), T(\text{id}_X), T(f))] = [(T(X), \text{id}_{T(X)}, T(f))] = Q(T(f)).$$

Hence we obtain $T_S Q = Q \circ T$.

We define a distinguished triangle in \mathcal{D}_S as a triangle in \mathcal{D}_S which is isomorphic to the image of a distinguished triangle in \mathcal{D} by Q .

If there is no risk of confusion, we often denote the automorphism Ts simply by T , and we will write a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ instead of a distinguished triangle $X \xrightarrow{Q(f)} Y \xrightarrow{Q(g)} Z \xrightarrow{Q(h)} TX$.

Definition 4.3. \mathcal{S} is said to be **compatible with the triangulation** if the following two axioms are satisfied:

MS6 $s \in \mathcal{S}$ if and only if $T(s) \in \mathcal{S}$.

MS7 Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'$ be distinguished triangles and let

$\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ be morphisms with $f' \circ \alpha = \beta \circ f$. Assume that α and β belong to \mathcal{S} . Then there exists a morphism $\gamma: Z \rightarrow Z'$ in \mathcal{S} giving rise to a morphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow T(\alpha) \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' \end{array}$$

Proposition 4.4. Let (\mathcal{D}, T) be a triangulated category and let \mathcal{S} be a multiplicative system in \mathcal{D} . Assume that \mathcal{S} is compatible with the triangulation. Then (\mathcal{D}_S, T_S) admits a unique structure of triangulated category such that $Q: \mathcal{D} \rightarrow \mathcal{D}_S$ is an exact functor.

Proof. (i) Let us check the axioms TR0–TR5 in Definition 3.2 of [OT14] in order to show that (\mathcal{D}_S, T_S) is a triangulated category.

(TR0) This follows from the definition of a distinguished triangle in \mathcal{D}_S .

(TR1) For all $X \in \mathcal{D}$, we have $Q(X) = X$ and $Q(\text{id}_X) = \text{id}_{Q(X)} = \text{id}_X$. Then by Lemma 4.2, we see that the axiom TR1 is satisfied.

(TR2) Let $\alpha = [(X', s, f)]$ be a morphism in \mathcal{D}_S with $f \in \text{Hom}_{\mathcal{D}}(X', Y)$ and $(s: X' \rightarrow X) \in \mathcal{S}$. Then by TR2, there exists a distinguished triangle $X' \xrightarrow{f} Y \rightarrow Z \rightarrow TX'$ in \mathcal{D} . Since $Q(s)$ is an isomorphism, we get the distinguished triangle $X \xrightarrow{s} Y \rightarrow Z \rightarrow TX$ in \mathcal{D}_S .

(TR3) This follows from Lemma 4.2.

(TR4) Consider two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'$ in \mathcal{D} and morphisms $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ in \mathcal{D}_S with $Q(f') \circ \alpha = \beta \circ Q(f)$. Then we have the following commutative diagram in \mathcal{D}_S :

$$\begin{array}{ccccccc} X & \xrightarrow{Q(f)} & Y & \xrightarrow{Q(g)} & Z & \xrightarrow{Q(h)} & TX \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow T(\alpha) \\ X' & \xrightarrow{Q(f')} & Y' & \xrightarrow{Q(g')} & Z' & \xrightarrow{Q(h')} & TX' \end{array}$$

By Proposition 4.8 of [Ter14] and the axioms TR2, TR4 and MS7, there exists a commutative diagram in \mathcal{D}

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z' & \xrightarrow{h} & TX \\
 \alpha' \uparrow & & \uparrow \beta' & & \uparrow \gamma' & & \uparrow T(\alpha') \\
 X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{\quad} & Z_1 & \xrightarrow{\quad} & TX_1 \\
 s \downarrow & & \downarrow t & & \downarrow u & & \downarrow T(s) \\
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'
 \end{array}$$

where the middle row is a distinguished triangle and $s, t, u \in \mathcal{S}$. Hence we obtain a morphism $\gamma : Z \rightarrow Z'$ and a commutative diagram in \mathcal{D}_s :

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\
 \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow T(\alpha) \\
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'
 \end{array}$$

(TR5) Consider two morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ in \mathcal{D}_s . Assume that $f = [(X, \text{id}_X, f')]$ and $g = [(Y, \text{id}_Y, g')]$. Then apply the axiom TR5 for f' and g' , and we get the octahedron diagram in \mathcal{D} . Hence its image by Q gives the octahedron diagram for f and g in \mathcal{D}_s .

(ii) By Lemma 4.2 and the definition of distinguished triangles in \mathcal{D}_s , $Q : \mathcal{D} \rightarrow \mathcal{D}_s$ is an exact functor.

(iii) It follows from Proposition 2.2.(ii) of [Ter14] that \mathcal{D}_s is unique up to equivalence of categories. By MS6, for any $s \in \mathcal{S}$, $T(s) \in \mathcal{S}$ and thus $Q(T(s))$ is an isomorphism. We have $T_s \circ Q = Q \circ T$ by Lemma 4.2, and thus by Proposition 2.2.(i) of [Ter14], T_s is unique up to unique isomorphism.

Let \mathcal{C} be a strictly full triangulated subcategory of (\mathcal{D}, T) . Then we define a family of morphisms $\mathcal{S}(\mathcal{C})$ in \mathcal{D} associated to \mathcal{C} as follows:

$$\mathcal{S}(\mathcal{C}) := \left\{ (f : X \rightarrow Y) \in \text{Mor}(\mathcal{D}) \left| \begin{array}{l} \text{there exists a distinguished triangle} \\ X \xrightarrow{f} Y \rightarrow Z \rightarrow TX. \\ \text{with } Z \in \mathcal{C} \end{array} \right. \right\}$$

Proposition 4.5. $\mathcal{S}(\mathcal{C})$ is a multiplicative system in \mathcal{D} , compatible with the triangulation.

Proof. Let us check the axioms MS1–RMS4 in Definition 3.1 of [Ter14].

(MS1) For any object $X \in \mathcal{D}$, the triangle $X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow TX$ is a distinguished triangle. Since $0 \in \mathcal{C}$, we have $\text{id}_X \in \mathcal{S}(\mathcal{C})$.

(MS2) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be in $\mathcal{S}(\mathcal{C})$. By TR3, there exist distinguished triangles $X \xrightarrow{f} Y \rightarrow Z' \rightarrow TX$, $Y \xrightarrow{g} Z \rightarrow X' \rightarrow TY$ and $X \xrightarrow{g \circ f} Z \xrightarrow{h} Y' \rightarrow TX$. By TR5, we obtain a distinguished triangle $Z' \rightarrow Y' \rightarrow X' \rightarrow TZ'$. Since $Z', X' \in \mathcal{C}$ and \mathcal{C} is extension-closed, $Y' \in \mathcal{C}$ and thus $g \circ f \in \mathcal{S}(\mathcal{C})$.

(RMS3) Similarly, let $f : Y \rightarrow X$ and $s : Y \rightarrow Y'$ be two morphisms with $s \in \mathcal{S}(\mathcal{C})$. Then there

exists a distinguished triangle $Z \xrightarrow{h} Y \xrightarrow{s} Y' \rightarrow TZ$ with $Z \in \mathcal{C}$. By TR2, there exists a distinguished triangle $Z \xrightarrow{f \circ h} X \xrightarrow{t} W \rightarrow TZ$, and by TR4, there exists a morphism of distinguished triangles:

$$\begin{array}{ccccccc} Z & \xrightarrow{h} & Y & \xrightarrow{s} & Y' & \longrightarrow & TZ \\ \parallel & & \downarrow f & & \downarrow g & & \parallel \\ Z & \xrightarrow{f \circ h} & X & \xrightarrow{t} & W & \longrightarrow & TZ. \end{array}$$

Since $Z \in \mathcal{C}$, we obtain $t \in \mathcal{S}(\mathcal{C})$.

(LMS3) Let $f : X \xrightarrow{t} Y$ and $s : Y' \rightarrow Y$ be two morphisms with $s \in \mathcal{S}(\mathcal{C})$. Then there exists a distinguished triangle $Y' \xrightarrow{s} Y \xrightarrow{h} Z \rightarrow TY'$ with $Z \in \mathcal{C}$. By TR2, there exists a distinguished triangle $W \rightarrow X \xrightarrow{h \circ f} Z \rightarrow TW$, and by Lemma 3.4 of [OT14], there exists a morphism of distinguished triangles:

$$\begin{array}{ccccccc} W & \xrightarrow{t} & X & \xrightarrow{h \circ f} & Z & \longrightarrow & TW \\ g \parallel & & \downarrow f & & \parallel & & \downarrow \\ Y' & \xrightarrow{s} & Y & \xrightarrow{h} & Z & \longrightarrow & TY'. \end{array}$$

Since $Z \in \mathcal{C}$, we obtain $t \in \mathcal{S}(\mathcal{C})$.

(RMS4 and LMS4) Let $f, g : X \rightarrow Y$ be two parallel morphisms in \mathcal{D} . Since \mathcal{D} is an additive category, it is enough to check that the following conditions are equivalent:

- (i) there exists $s : Y \rightarrow Y'$ in $\mathcal{S}(\mathcal{C})$ with $s \circ f = 0$.
- (ii) there exists $t : X' \rightarrow X$ in $\mathcal{S}(\mathcal{C})$ with $f \circ t = 0$.

(i) \Rightarrow (ii). If $s : Y \rightarrow Y'$ in $\mathcal{S}(\mathcal{C})$, then by TR3, there exists a distinguished triangle $T^{-1}W \xrightarrow{v} Y \xrightarrow{s} Y' \rightarrow W$ with $W \in \mathcal{C}$. Since $s \circ f = 0$, by Proposition 3.8 of [OT14], there exists a morphism $h : X \rightarrow T^{-1}W$ such that $f = v \circ h$. Then, by TR2 and TR3, we obtain a distinguished triangle $X' \xrightarrow{t} X \xrightarrow{h} T^{-1}W \rightarrow TX'$ and a morphism $t : X' \rightarrow X \in \mathcal{S}(\mathcal{C})$ since $W \in \mathcal{C}$. By Proposition 3.5 of [OT14], we get $f \circ t = v \circ h \circ t = 0$.

(ii) \Rightarrow (i). Since $t : X' \rightarrow X$ in $\mathcal{S}(\mathcal{C})$, there exists a distinguished triangle $X' \xrightarrow{t} X \xrightarrow{u} Z \rightarrow TX'$ with $Z \in \mathcal{C}$. Since $f \circ t = 0$, by Proposition 3.8 of [OT14], there exists a morphism $g : Z \rightarrow Y$ such that $f = g \circ u$. Then, by TR2, we obtain a distinguished triangle $Z \xrightarrow{g} Y \xrightarrow{s} Y' \rightarrow TZ$ and a morphism $s : Y \rightarrow Y' \in \mathcal{S}(\mathcal{C})$ since $Z \in \mathcal{C}$. By Proposition 3.5 of [OT14], we get $s \circ f = s \circ g \circ u = 0$.

(MS 6) If $(s : X \rightarrow Y) \in \mathcal{S}(\mathcal{C})$, there exists a distinguished triangle $X \xrightarrow{s} Y \xrightarrow{g} Z \xrightarrow{h} TX$ in \mathcal{D} , and by TR3, we have a distinguished triangle $TX \xrightarrow{-T(s)} TY \xrightarrow{-T(g)} TZ \xrightarrow{-T(h)} T^2X$. From the isomorphism of distinguished triangles:

$$\begin{array}{ccccccc} TX & \xrightarrow{-T(s)} & TY & \xrightarrow{-T(g)} & TZ & \xrightarrow{-T(h)} & T^2X \\ \parallel & & \downarrow -\text{id}_T & & \parallel & & \parallel \\ TX & \xrightarrow{T(s)} & TY & \xrightarrow{T(g)} & TZ & \xrightarrow{-T(h)} & T^2X \end{array}$$

we get $T(s) \in \mathcal{S}(\mathcal{C})$ because $Z \in \mathcal{C}$. Conversely, if $T(s) \in \mathcal{S}(\mathcal{C})$, we get $s \in \mathcal{S}(\mathcal{C})$ in the same

way.

(MS7) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'$ be distinguished triangles and let $\alpha : X \rightarrow X'$ and $\beta : Y \rightarrow Y'$ be morphisms with $f' \circ \alpha = \beta \circ f$. Assume that α and β belong to $\mathcal{S}(\mathcal{C})$. By Proposition 3.20 of [OT14], we get the following diagram of distinguished triangles in \mathcal{D} :

$$\begin{array}{ccccccc}
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\
 \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow T(\alpha) \\
 X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X'' & \dashrightarrow & Y'' & \dashrightarrow & Z'' & \dashrightarrow & TX'' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 TX & \longrightarrow & TY & \longrightarrow & TZ & \longrightarrow & T^2X.
 \end{array}$$

ac

Since $\alpha, \beta \in \mathcal{S}(\mathcal{C})$, $X'', Y'' \in \mathcal{C}$, and thus $Z'' \in \mathcal{C}$. Hence we obtain $\gamma \in \mathcal{S}(\mathcal{C})$. □

Therefore we can obtain the localization of a triangulated category (\mathcal{D}, T) by a multiplicative system $\mathcal{S}(\mathcal{C})$.

Definition 4.6. The localization $\mathcal{D}_{\mathcal{S}(\mathcal{C})}$ of \mathcal{D} by $\mathcal{S}(\mathcal{C})$ is called the **Verdier quotient** of \mathcal{D} by \mathcal{C} and denoted by \mathcal{D}/\mathcal{C} . The localization functor $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$ is called the **Verdier localization functor**. □

Definition 4.7. A subcategory \mathcal{C} of a triangulated category (\mathcal{D}, T) is said to be **epaisse** if it is a full triangulated subcategory of \mathcal{D} and the following condition EP is satisfied:

EP If $X \rightarrow Y$ is a morphism in \mathcal{D} which is contained in a distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow TX$ with $Z \in \mathcal{C}$, and if the morphism also factors through an object W of \mathcal{C} , then X and Y are objects of \mathcal{C} . □

Definition 4.8. A full subcategory \mathcal{C} of a triangulated category \mathcal{D} is **closed under direct summands** if every object of \mathcal{D} which is a direct summand of an object of \mathcal{C} is itself an object of \mathcal{C} . □

Remark 4.9. From definitions, we have the following.

1. An epaisse subcategory \mathcal{C} of a triangulated category \mathcal{D} is strictly full.
2. A full subcategory \mathcal{C} of a triangulated category \mathcal{D} which is closed under direct summands is strictly full. □

Proposition 4.10. A full triangulated subcategory \mathcal{C} of a triangulated category \mathcal{D} is epaisse if and only if it is closed under direct summands.

Proof. Suppose first that \mathcal{C} is epaisse and that $X \simeq X_1 \oplus X_2$ is in \mathcal{C} . Then we have a

distinguished triangle $X_1 \rightarrow X \rightarrow X_2 \xrightarrow{0} TX_1$. Hence we obtain a distinguished triangle

$$\begin{array}{ccccc} TX_2 & \xrightarrow{0} & X_1 & \rightarrow & X \rightarrow X_2, \\ & \searrow & \nearrow & & \\ & & 0 & & \end{array}$$

so X_1 and X_2 belong to \mathcal{C} .

Suppose now that \mathcal{C} is closed under direct summands, and let $f : X \rightarrow Y$ be a morphism in \mathcal{D} that factors through an object W of \mathcal{C} and that is contained in a distinguished triangle

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} TX \\ & \searrow \alpha & \nearrow \beta & & \\ & & W & & \end{array}$$

with $Z \in \mathcal{C}$. Then we have the following diagram by the octahedron axiom for $X \xrightarrow{\alpha} W \xrightarrow{\beta} Y$:

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & W & \xrightarrow{\gamma} & L & \xrightarrow{\gamma} & TX \\ \text{id} \downarrow & & \downarrow \beta & & \downarrow & & \downarrow \text{id} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \alpha \downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow T(\alpha) \\ W & \xrightarrow{\beta} & Y & \xrightarrow{\quad} & M & \xrightarrow{\quad} & TW \\ \downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow \\ L & \dashrightarrow & Z & \dashrightarrow & M & \dashrightarrow & TL. \end{array}$$

Consider the composition of morphisms $Z \xrightarrow{h} TX \xrightarrow{T(\alpha)} TW$ coming from this diagram. Then we obtain the following diagram by the octahedron axiom:

$$\begin{array}{ccccccc} Z & \xrightarrow{h} & TX & \xrightarrow{-T(f)} & TY & \xrightarrow{\quad} & TZ \\ \text{id} \downarrow & & \downarrow T(\alpha) & & \downarrow & & \downarrow \text{id} \\ Z & \xrightarrow{T(\alpha) \circ h} & TW & \xrightarrow{g} & N & \xrightarrow{h} & TZ \\ h \downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow \\ TX & \xrightarrow{T(\alpha)} & TW & \xrightarrow{\quad} & TL & \xrightarrow{-T(\gamma)} & T^2 X \\ -T(f) \downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow -T^2(f) \\ TY & \dashrightarrow & N & \dashrightarrow & TL & \dashrightarrow & T^2 Y. \end{array}$$

The morphism $L \xrightarrow{T(f) \circ \gamma} TY$ in this diagram factors as

$$L \xrightarrow{\gamma} TX \xrightarrow{T(\alpha)} TW \xrightarrow{T(\beta)} TY,$$

but the composition $L \xrightarrow{\gamma} TX \xrightarrow{T(\alpha)} TW$ is zero, since it is the composition of two morphisms in a distinguished triangle. Hence N is isomorphic to the direct sum $TY \oplus TL$, thus Y is an object of \mathcal{C} , and so is X . \square

Definition 4.11. Let (\mathcal{D}, T) and (\mathcal{D}', T') be triangulated categories and let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be

an exact functor. The **kernel** of F is the full subcategory \mathcal{C} of \mathcal{D} whose objects map to objects of \mathcal{D}' isomorphic to 0, i.e.,

$$\text{Ob}(\mathcal{C}) = \{X \in \text{Ob}(\mathcal{D}) \mid F(X) \simeq 0\}.$$

The kernel of F is denoted by $\text{Ker}(F)$.

Proposition 4.12. *Let (\mathcal{D}, T) and (\mathcal{D}', T') be triangulated categories and let $F: \mathcal{D} \Rightarrow \mathcal{D}'$ be an exact functor. Then the kernel of F is an epaisse subcategory of \mathcal{D} .*

Proof. Since F is exact, we have $FT(X) \simeq TF(X)$ and $T^{-1}F(X) \simeq FT^{-1}(X)$ for any $X \in \mathcal{D}$. Then we obtain $TX \in \text{Ker}(F)$ and $T^{-1}X \in \text{Ker}(F)$ for any $X \in \text{Ker}(F)$. Hence $\text{Ker}(F)$ is closed under translation.

Let $X \rightarrow Y \rightarrow Z \rightarrow TX$ be a distinguished triangle in \mathcal{D} with $X, Y \in \text{Ker}(F)$. Then $F(Z) \simeq 0$, since F is exact. Hence $Z \in \text{Ker}(F)$. Therefore $\text{Ker}(F)$ is a triangulated subcategory of \mathcal{D} .

Since F is an additive functor, we have $F(X \oplus Y) = F(X) \oplus F(Y)$. If $F(X \oplus Y) \simeq 0$, then so are $F(X)$ and $F(Y)$, since they are direct summands of 0. Hence $\text{Ker}(F)$ is closed under direct summands. \square

Proposition 4.13. *Let \mathcal{C} be a strictly full triangulated subcategory of a triangulated category (\mathcal{D}, T) . Let \mathcal{D}/\mathcal{C} be the Verdier quotient of \mathcal{D} by \mathcal{C} and $Q: \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$ the Verdier localization functor.*

(i) *If $X \in \mathcal{C}$, then $Q(X) \simeq 0$. Hence $\mathcal{C} \subseteq \text{Ker}(Q)$.*

(ii) *Assume that \mathcal{C} is epaisse. Then $X \in \mathcal{C}$ if and only if $Q(X) \simeq 0$. Hence $\mathcal{C} = \text{Ker}(Q)$.*

Proof. (i) Consider a distinguished triangle $0 \rightarrow X \xrightarrow{\text{id}_X} X \rightarrow T(0)$. Since $X \in \mathcal{C}$, the morphism $0 \rightarrow X$ belongs to $\mathcal{S}(\mathcal{C})$, and thus $Q(0) \rightarrow Q(X)$ is an isomorphism. $Q(0) = 0$ because Q is an additive functor. Hence $Q(X)$ is isomorphic to 0.

(ii) If $Q(X) \simeq 0$, then by Proposition 3.2, there exists $Y \in \mathcal{D}$ such that $0: Y \rightarrow X$ belongs to $\mathcal{S}(\mathcal{C})$. Hence we have a distinguished triangle $X \rightarrow Z \rightarrow TY \xrightarrow{0} TX$ with $Z \in \mathcal{C}$. Then $Z \simeq X \oplus TY$. Since \mathcal{C} is epaisse, we obtain $X \in \mathcal{C}$. \square

Proposition 4.14. *Let $F: \mathcal{D} \Rightarrow \mathcal{D}'$ be an exact functor of triangulated categories. Assume that \mathcal{C} is a triangulated subcategory of \mathcal{D} such that $\mathcal{C} \subseteq \text{Ker}(F)$. Then F factors uniquely through the Verdier quotient $Q: \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$.*

Proof. Let $s: X \rightarrow Y$ be any morphism in $\mathcal{S}(\mathcal{C})$. Then there exists a distinguished triangle $X \xrightarrow{s} Y \rightarrow Z \rightarrow TX$ with $Z \in \mathcal{C}$. Apply the exact functor F to this triangle, and we get a distinguished triangle

$$F(X) \xrightarrow{F(s)} F(Y) \rightarrow F(Z) \rightarrow TF(X).$$

By Corollary 3.12 of [OT14], $F(s)$ is an isomorphism since $F(Z) \simeq 0$. Hence it follows from the axiom L2 that F factors uniquely through Q . \square

Proposition 4.15. *If \mathcal{C} is an epaisse subcategory of a triangulated category \mathcal{D} , then the multiplicative system $\mathcal{S}(\mathcal{C})$ associated with \mathcal{C} is saturated.*

Proof. Let $r : V \rightarrow X$, $s : X \rightarrow Y$ and $t : Y \rightarrow Z$ be morphisms such that $s \circ r$ and $t \circ s$ belong to $\mathcal{S}(\mathcal{C})$. Suppose that morphisms $s \circ r$, s , $t \circ s$ and t are completed to distinguished triangles

$$\begin{aligned} V &\xrightarrow{s \circ r} Y \rightarrow W \rightarrow TV, \\ X &\xrightarrow{s} Y \xrightarrow{g} Z' \rightarrow TX, \\ X &\xrightarrow{t \circ s} Z \rightarrow Y' \rightarrow TX, \\ Y &\xrightarrow{t} Z \rightarrow X' \xrightarrow{f} TY. \end{aligned}$$

Since $g \circ (s \circ r) = (g \circ s) \circ t = 0$, g factors through an object W of \mathcal{C} because we obtain the following diagram of distinguished triangles

$$\begin{array}{ccccccc} V & \xrightarrow{s \circ r} & Y & \longrightarrow & W & \longrightarrow & TV \\ 0 \downarrow & & \downarrow g & & \downarrow & & \downarrow T(0) \\ 0 & \longrightarrow & Z' & \xrightarrow{\text{id}_{Z'}} & Z' & \longrightarrow & 0. \end{array}$$

Then $T(g) \circ f : X' \rightarrow TZ'$ also factors through TW .

Now we have the following diagram by the octahedron axiom:

$$\begin{array}{ccccccc} X & \xrightarrow{s} & Y & \xrightarrow{g} & Z' & \longrightarrow & TX \\ \text{id} \downarrow & & \downarrow t & & \downarrow & & \downarrow \text{id} \\ X & \xrightarrow{t \circ s} & Z & \longrightarrow & Y' & \longrightarrow & TX \\ s \downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow T(s) \\ Y & \xrightarrow{t} & Z & \longrightarrow & X' & \xrightarrow{f} & TY \\ \downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow \\ Z' & \dashrightarrow & Y' & \dashrightarrow & X' & \dashrightarrow & TZ'. \end{array}$$

$T(g) \circ f$

Since $t \circ s \in \mathcal{S}(\mathcal{C})$, we have $Y' \in \mathcal{C}$, and $TY' \in \mathcal{C}$. Thus we see X' and TZ' are objects of \mathcal{C} because we have the following diagram

$$X' \xrightarrow{T(g) \circ f} TZ' \rightarrow TY' \rightarrow TX'.$$

In particular, Z' is an object of \mathcal{C} . Therefore s is in $\mathcal{S}(\mathcal{C})$. □

Definition 4.16. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be exact functors between triangulated categories \mathcal{C} , \mathcal{D} and \mathcal{E} , respectively. Then a sequence $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is called an **exact triple** if

- (i) \mathcal{C} is an epaisse subcategory of \mathcal{D} and F is the inclusion functor,
- (ii) \mathcal{E} is the Verdier quotient \mathcal{D}/\mathcal{C} of \mathcal{D} by \mathcal{C} and G is the Verdier localization functor. □

5 Admissible subcategories

Let \mathcal{D} be an additive category and let \mathcal{A} be a full subcategory of \mathcal{D} . Then we have the

following:

1. \mathcal{A}^\perp and ${}^\perp\mathcal{A}$ are strictly full additive subcategories of \mathcal{D} .
2. $\mathcal{A} \cap \mathcal{A}^\perp = 0$ and $\mathcal{A} \cap {}^\perp\mathcal{A} = 0$.

Proposition 5.1. *If (\mathcal{D}, T) is a triangulated category and \mathcal{A} is a full triangulated subcategory of \mathcal{D} , then \mathcal{A}^\perp and ${}^\perp\mathcal{A}$ are also triangulated subcategories.*

Proof. Let $Y \in \mathcal{A}^\perp$. Since $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in \mathcal{A}$, we have

$$\text{Hom}_{\mathcal{D}}(X, T^i Y) \simeq \text{Hom}_{\mathcal{D}}(T^{-i} X, Y) = 0$$

for all $X \in \mathcal{A}$ and all $i \in \mathbb{Z}$. Hence $T^i Y \in \mathcal{A}^\perp$, and thus $T\mathcal{A}^\perp = \mathcal{A}^\perp$.

Let $X \rightarrow Y \rightarrow Z \rightarrow TX$ be a distinguished triangle in \mathcal{D} with $X, Y \in \mathcal{A}^\perp$. Then applying $\text{Hom}_{\mathcal{D}}(W, \bullet)$ with any $W \in \mathcal{A}$, we have the exact sequence

$$\text{Hom}(W, Y) \rightarrow \text{Hom}(W, Z) \rightarrow \text{Hom}(W, TX).$$

Since $\text{Hom}(W, Y) = \text{Hom}(W, TX) = 0$, we obtain $\text{Hom}(W, Z) = 0$, and thus $Z \in \mathcal{A}^\perp$. By Corollary 3.19 of [OT 14], we see that \mathcal{A}^\perp is a triangulated category.

Similarly, let $X \in {}^\perp\mathcal{A}$. Since $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $Y \in \mathcal{A}$, we have

$$\text{Hom}(T^i X, Y) \simeq \text{Hom}(X, T^{-i} Y) = 0$$

for all $Y \in \mathcal{A}$ and all $i \in \mathbb{Z}$. Hence $T^i X \in {}^\perp\mathcal{A}$, and thus $T({}^\perp\mathcal{A}) = {}^\perp\mathcal{A}$.

Let $X \rightarrow Y \rightarrow Z \rightarrow TX$ be a distinguished triangle in \mathcal{D} with $X, Y \in {}^\perp\mathcal{A}$. Then applying $\text{Hom}_{\mathcal{D}}(\bullet, W)$ with any $W \in \mathcal{A}$, we have the exact sequence

$$\text{Hom}(TX, W) \rightarrow \text{Hom}(Z, W) \rightarrow \text{Hom}(Y, W).$$

Since $\text{Hom}(TX, W) = \text{Hom}(Y, W) = 0$, we obtain $\text{Hom}(Z, W) = 0$, and thus $Z \in {}^\perp\mathcal{A}$. By Corollary 3.19 of [OT14], we see that ${}^\perp\mathcal{A}$ is a triangulated category. \square

The category \mathcal{A} is contained in the second orthogonals ${}^\perp(\mathcal{A}^\perp)$ and $({}^\perp\mathcal{A})^\perp$. If it is necessary to indicate the ambient category \mathcal{D} , the orthogonals will be denoted as, e.g., $(\mathcal{A}^\perp)\mathcal{D}$.

Lemma 5.2. *Let (\mathcal{D}, T) be a triangulated category and let \mathcal{A} be a strictly full triangulated subcategory of \mathcal{D} . Then we have the followings:*

- (i) For any $X \in \mathcal{D}$ and any $Y \in \mathcal{A}^\perp$,

$$\text{Hom}_{\mathcal{D}}(X, {}^\iota Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}/\mathcal{A}}(QX, Q{}^\iota Y),$$

where ${}^\iota : \mathcal{A}^\perp \rightarrow \mathcal{D}$ the inclusion functor. In particular, the composite functor $Q{}^\iota : \mathcal{A}^\perp \rightarrow \mathcal{D}/\mathcal{A}$ is fully faithful.

- (ii) For any $X \in \mathcal{D}$ and any $Y \in {}^\perp\mathcal{A}$,

$$\text{Hom}_{\mathcal{D}}({}^\perp Y, X) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}/\mathcal{A}}(Q{}^\perp Y, QX),$$

where ${}^\perp : {}^\perp\mathcal{A} \rightarrow \mathcal{D}$ the inclusion functor. In particular, the composite functor $Q \circ {}^\perp : {}^\perp\mathcal{A} \rightarrow \mathcal{D}/\mathcal{A}$ is fully faithful.

Proof. We give only the proof of (i), since the proof of (ii) is similar to that of (i).

Let $(s : X' \rightarrow X) \in \mathcal{S}(\mathcal{A})$. Then we have a distinguished triangle $X' \xrightarrow{s} X \rightarrow X'' \rightarrow TX'$ with

- (i) Assume that F admits a right adjoint functor G . The unit η gives a distinguished triangle

$$FGY \xrightarrow{\eta_Y} Y \rightarrow \text{Cone}(\eta_Y) \rightarrow TFGY$$

for any $Y \in \mathcal{D}$. Then $\text{Hom}_{\mathcal{D}}(FX, \text{Cone}(\eta_Y)) = 0$ for all $X \in \mathcal{C}$ and all $Y \in \mathcal{D}$.

- (ii) Assume that F admits a left adjoint functor H . The counit ε gives a distinguished triangle

$$Y \xrightarrow{\varepsilon_Y} FHY \rightarrow \text{Cone}(\varepsilon_Y) \rightarrow TY$$

for any $Y \in \mathcal{D}$. Then $\text{Hom}_{\mathcal{D}}(\text{Cone}(\varepsilon_Y), FX) = 0$ for all $X \in \mathcal{C}$ and all $Y \in \mathcal{D}$.

Proof. We give only the proof of (i), since the proof of (ii) is similar to that of (i).

Applying the functor $\text{Hom}_{\mathcal{D}}(FX, \bullet)$ with $X \in \mathcal{C}$ to the distinguished triangle above, we have a long exact sequence

$$\begin{aligned} \cdots &\rightarrow \text{Hom}_{\mathcal{D}}(FX, FGY) \xrightarrow{\eta_Y^*} \text{Hom}_{\mathcal{D}}(FX, Y) \rightarrow \text{Hom}_{\mathcal{D}}(FX, \text{Cone}(\eta_Y)) \\ &\rightarrow \text{Hom}_{\mathcal{D}}(FX, FGTY) \xrightarrow{-T(\eta_Y)^* = -\eta_{TY}^*} \text{Hom}_{\mathcal{D}}(FX, TY) \rightarrow \cdots \end{aligned}$$

Then it follows from Proposition 2.3 that $\Phi_{X,Y}(\eta_Y \circ F(f)) = f$ for any $f \in \text{Hom}_{\mathcal{C}}(X, GY)$ and the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(FX, FGY) & \xrightarrow{\eta_Y^*} & \text{Hom}_{\mathcal{D}}(FX, Y) \\ \uparrow \wr F & & \downarrow \wr \Phi_{X,Y} \\ \text{Hom}_{\mathcal{C}}(X, GY) & \xlongequal{\quad} & \text{Hom}_{\mathcal{C}}(X, GY) \end{array}$$

Hence η_Y^* is an isomorphism. Similarly, $-T(\eta_Y)^*$ is also an isomorphism. Therefore we obtain $\text{Hom}_{\mathcal{D}}(FX, \text{Cone}(\eta_Y)) = 0$. \square

Proposition 5.8. Let (\mathcal{D}, T) be a triangulated category and let \mathcal{A} be a strictly full triangulated subcategory of \mathcal{D} . Suppose that $\iota : \mathcal{A} \rightarrow \mathcal{D}$ and $\iota^\perp : \mathcal{A}^\perp \rightarrow \mathcal{D}$ are the inclusion functors and $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{A}$ is the Verdier localization functor. Then the following conditions are equivalent:

- (i) \mathcal{A} is right admissible.
- (ii) For each $X \in \mathcal{D}$, there exists a distinguished triangle

$$X_{\mathcal{A}} \rightarrow X \rightarrow X_{\mathcal{A}^\perp} \rightarrow TX_{\mathcal{A}}$$

with $X_{\mathcal{A}} \in \mathcal{A}$ and $X_{\mathcal{A}^\perp} \in \mathcal{A}^\perp$.

- (iii) \mathcal{A} is an epaisse subcategory in \mathcal{D} and the composition functor $Q \circ \iota^\perp : \mathcal{A}^\perp \rightarrow \mathcal{D}/\mathcal{A}$ is an exact equivalence.
- (iv) \mathcal{A} is an epaisse subcategory in \mathcal{D} and Q admits a right adjoint functor Q_* .
- (v) \mathcal{D} is generated by \mathcal{A} and \mathcal{A}^\perp as a triangulated category.

In this case, we have a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$, and such a triangle in (ii) is unique up to unique isomorphism.

Proof. (i) \Rightarrow (ii). The unit $\eta : \iota \circ \iota^\perp \rightarrow \text{id}_{\mathcal{D}}$ of the adjunction gives a distinguished triangle in \mathcal{D}

$$\iota \iota^\perp X \xrightarrow{\eta_X} X \rightarrow Y \rightarrow T \iota \iota^\perp X$$

for any $X \in \mathcal{D}$. By Lemma 5.7, $\text{Hom}_{\mathcal{D}}(\iota Z, Y) = 0$ for any $Z \in \mathcal{A}$. Hence $Y \in \mathcal{A}^\perp$. Therefore we obtain the required distinguished triangle with $X_{\mathcal{A}} := \iota \iota^\perp X \in \mathcal{A}$ and $X_{\mathcal{A}^\perp} := Y \in \mathcal{A}^\perp$.

(ii) \Rightarrow (i). Let $X \in \mathcal{D}$. Then there exists a distinguished triangle

$$Y \rightarrow X \rightarrow Z \rightarrow TY$$

with $Y \in \mathcal{A}$ and $Z \in \mathcal{A}^\perp$. Then we define $\iota^! X := Y$.

Let us show that such a triangle is unique up to unique isomorphism and the correspondence extends to a functor. To do this, let $Y' \rightarrow X' \rightarrow Z' \rightarrow TY'$ be another such triangle with $Y' \in \mathcal{A}$ and $Z' \in \mathcal{A}^\perp$ and let $f : X \rightarrow X'$ be a morphism. We show that there exists a unique $\psi : Y \rightarrow Y'$ making the following diagram commutative.

$$\begin{array}{ccccccc} Y & \xrightarrow{u} & X & \longrightarrow & Z & \longrightarrow & TY \\ \psi \downarrow & & \downarrow f & & & & \\ Y' & \longrightarrow & X' & \longrightarrow & Z' & \longrightarrow & TY' \end{array}$$

Applying the functor $\text{Hom}_{\mathcal{D}}(Y, \bullet)$ to the second distinguished triangle, and taking into account that $\text{Hom}_{\mathcal{D}}(Y, T^{-1}Z') = 0$ and $\text{Hom}_{\mathcal{D}}(Y, Z') = 0$, we see that $\text{Hom}_{\mathcal{D}}(Y, Y') \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(Y, X')$. In that case, ψ is the preimage of $f \circ u$ under this isomorphism. This means that the correspondence extends to a functor uniquely. In particular, if we set $X' = X$ in the diagram and if f is the identity morphism id_X , then we obtain a unique isomorphism as ψ . Therefore the correspondence $\iota^! : \mathcal{D} \rightarrow \mathcal{A}$ is a functor.

(ii) \Rightarrow (iii). We show that \mathcal{A} is an epaisse subcategory in \mathcal{D} . Let X and Y be objects of \mathcal{D} . Assume $X \oplus Y \in \mathcal{A}$. We have the following two distinguished triangles.

$$\begin{aligned} X_{\mathcal{A}} \rightarrow X \rightarrow X_{\mathcal{A}^\perp} \rightarrow TX_{\mathcal{A}}, \\ Y_{\mathcal{A}} \rightarrow Y \rightarrow Y_{\mathcal{A}^\perp} \rightarrow TY_{\mathcal{A}}, \end{aligned}$$

where $X_{\mathcal{A}} \rightarrow Y_{\mathcal{A}} \in \mathcal{A}$ and $X_{\mathcal{A}^\perp}, Y_{\mathcal{A}^\perp} \in \mathcal{A}^\perp$, respectively. Then we obtain the distinguished triangle

$$X_{\mathcal{A}} \oplus Y_{\mathcal{A}} \rightarrow X \oplus Y \rightarrow X_{\mathcal{A}^\perp} \oplus Y_{\mathcal{A}^\perp} \rightarrow TX_{\mathcal{A}} \oplus TY_{\mathcal{A}}.$$

Since both $X_{\mathcal{A}} \oplus Y_{\mathcal{A}}$ and $X \oplus Y$ are object of \mathcal{A} , $X_{\mathcal{A}^\perp} \oplus Y_{\mathcal{A}^\perp}$ is also an object of \mathcal{A} because \mathcal{A} is strictly full. So $X_{\mathcal{A}^\perp} \oplus Y_{\mathcal{A}^\perp} \in \mathcal{A} \cap \mathcal{A}^\perp$, and then $X_{\mathcal{A}^\perp} \oplus Y_{\mathcal{A}^\perp} = 0$. Hence $X_{\mathcal{A}^\perp} = 0$ and $Y_{\mathcal{A}^\perp} = 0$. Therefore we obtain $X \simeq X_{\mathcal{A}}$ and $Y \simeq Y_{\mathcal{A}}$, and thus X and Y are objects of \mathcal{A} because \mathcal{A} is strictly full.

Next we show that the functor $Q \circ \iota^\perp$ is an exact equivalence. In order to show this, we prove that $Q \circ \iota^\perp$ is essentially surjective, because $Q \circ \iota^\perp$ is fully faithful by Lemma 5.2. For any $Y = QX \in \mathcal{D}/\mathcal{A}$ with $X \in \mathcal{D}$, we have a distinguished triangle

$$X_{\mathcal{A}} \rightarrow X \rightarrow X_{\mathcal{A}^\perp} \rightarrow TX_{\mathcal{A}}$$

with $X_{\mathcal{A}} \in \mathcal{A}$ and $X_{\mathcal{A}^\perp} \in \mathcal{A}^\perp$. Then we obtain $Y \simeq Q(X_{\mathcal{A}^\perp})$.

(iii) \Rightarrow (ii). For any $X \in \mathcal{D}$, there exists an object $Y \in \mathcal{A}^\perp$ such that $\bar{\theta} : QX \rightarrow Q\iota^\perp Y$ is an isomorphism, because $Q \circ \iota^\perp$ is essentially surjective. By Lemma 5.2, we obtain a morphism $\theta : X \rightarrow \iota^\perp Y$ with $Q(\theta) = \bar{\theta}$. Then we have a distinguished triangle

$$Z \rightarrow X \xrightarrow{\theta} \iota^\perp Y \rightarrow TZ.$$

Hence $QZ \simeq 0$, and $Z \in \mathcal{A}$. Therefore we obtain the required distinguished triangle by setting $X_{\mathcal{A}} := Z$ and $X_{\mathcal{A}^\perp} := \iota^\perp Y$.

(iii) \Rightarrow (iv). Put $p := Q \circ \iota^\perp$ and $Q_* := \iota^\perp \circ q$, where q is the quasi-inverse to p . Then we have

$$Q \circ Q_* = Q \circ (\iota^\perp \circ q) = p \circ q \simeq \text{id}_{\mathcal{D}/\mathcal{A}}.$$

For any $Y \in \mathcal{D}$ and any $Z \in \mathcal{D}/\mathcal{A}$, since $Q_*Z = \iota^\perp qZ$, by Lemma 5.2, we have

$$\text{Hom}_{\mathcal{D}}(Y, Q_*Z) \simeq \text{Hom}_{\mathcal{D}/\mathcal{A}}(QY, QQ_*Z) \simeq \text{Hom}_{\mathcal{D}/\mathcal{A}}(QY, Z).$$

Hence Q has a right adjoint functor Q_* .

(iv) \Rightarrow (iii). For any $Y \in \mathcal{A}$ and any $Z \in \mathcal{D}/\mathcal{A}$, we have

$$\text{Hom}_{\mathcal{D}}(\iota Y, Q_*Z) \simeq \text{Hom}_{\mathcal{D}/\mathcal{A}}(Q\iota Y, Z) = 0$$

since $Q \circ \iota = 0$. Hence Q_* factors through \mathcal{A}^\perp as follows.

$$\begin{array}{ccc} \mathcal{D}/\mathcal{A} & \xrightarrow{q} & \mathcal{A}^\perp \\ & \searrow Q_* & \downarrow \iota^\perp \\ & & \mathcal{D}. \end{array}$$

For any $Y \in \mathcal{A}^\perp$ and any $Z \in \mathcal{D}/\mathcal{A}$, we have

$$\text{Hom}_{\mathcal{D}/\mathcal{A}}(pY, Z) \simeq \text{Hom}_{\mathcal{D}}(Q\iota^\perp Y, Z) \simeq \text{Hom}_{\mathcal{D}}(\iota^\perp Y, Q_*Z) \simeq \text{Hom}_{\mathcal{A}^\perp}(Y, qZ).$$

Hence q is a right adjoint functor to p . Since p is fully faithful by Lemma 5.2, we see that the counit $\varepsilon' : \text{id}_{\mathcal{A}^\perp} \Rightarrow q \circ p$ is an isomorphism from Proposition 2.8.

Next we show that the unit $\eta' : p \circ q \Rightarrow \text{id}_{\mathcal{D}/\mathcal{A}}$ is an isomorphism. For any $X \in \mathcal{D}/\mathcal{A}$, we consider a distinguished triangle

$$pqX \xrightarrow{\eta'_X} X \rightarrow Y \rightarrow TpqX$$

with $Y \in \mathcal{D}/\mathcal{A}$. By Lemma 5.7, $\text{Hom}_{\mathcal{A}^\perp}(Z, qY) = \text{Hom}_{\mathcal{D}/\mathcal{A}}(pZ, Y) = 0$ for any $Z \in \mathcal{A}^\perp$. Hence $qY \simeq 0$, and $Q_*Y \simeq 0$. For any $W \in \mathcal{D}/\mathcal{A}$,

$$\text{Hom}_{\mathcal{D}/\mathcal{A}}(W, Y) = \text{Hom}_{\mathcal{D}/\mathcal{A}}(QW, Y) \simeq \text{Hom}_{\mathcal{D}}(W, Q_*Y) = 0.$$

Therefore We obtain $Y \simeq 0$, and thus the unit η' is an isomorphism.

(ii) \Rightarrow (v). Clear.

(v) \Rightarrow (ii). We consider the full subcategory \mathcal{D}' generated by the objects $X \in \mathcal{D}$ which can be included in the distinguished triangle $Y \rightarrow X \rightarrow Z \rightarrow TY$ with $Y \in \mathcal{A}$ and $Z \in \mathcal{A}^\perp$. If \mathcal{D}' is a triangulated subcategory of \mathcal{D} , then we have $\mathcal{D}' = \mathcal{D}$ from the condition (v), and we obtain the condition (ii).

First we show that \mathcal{D}' is closed under translation. For any object $X \in \mathcal{D}'$, we have a distinguished triangle $Y \rightarrow X \rightarrow Z \rightarrow TY$ with $Y \in \mathcal{A}$ and $Z \in \mathcal{A}^\perp$. Then for any $i \in \mathbb{Z}$, we obtain a distinguished triangle $T^iY \rightarrow T^iX \rightarrow T^iZ \rightarrow T^{i+1}Y$. Since \mathcal{A} and \mathcal{A}^\perp are triangulated categories, $T^iY \in \mathcal{A}$ and $T^iZ \in \mathcal{A}^\perp$, and thus $T^iX \in \mathcal{D}'$.

Next we show that \mathcal{D}' is closed under taking a cone of a morphism. Let X, X' be object of \mathcal{D}' and $f : X \rightarrow X'$ be a morphism. Then we have two distinguished triangles $Y \xrightarrow{u} X \xrightarrow{v} Z \xrightarrow{w} TY$ with $Y \in \mathcal{A}$ and $Z \in \mathcal{A}^\perp$ and $Y' \xrightarrow{u'} X' \xrightarrow{v'} Z' \xrightarrow{w'} TY'$ with $Y' \in \mathcal{A}$ and $Z' \in \mathcal{A}^\perp$.

We prove that there exist morphisms $\phi : Z \rightarrow Z'$ and $\psi : Y \rightarrow Y'$ giving a morphism of distinguished triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{u} & X & \xrightarrow{v} & Z & \xrightarrow{w} & TY \\ \psi \downarrow & & \downarrow f & & \downarrow \phi & & \downarrow T(\psi) \\ Y' & \xrightarrow{u'} & X' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TY' \end{array}$$

Applying the functor $\text{Hom}_{\mathcal{D}}(\bullet, Z')$ to the first distinguished triangle, and taking into account that $\text{Hom}_{\mathcal{D}}(Y, Z') = 0$, we see that $v^* : \text{Hom}_{\mathcal{D}}(Z, Z') \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(X, Z')$. Hence we obtain $\phi : Z \rightarrow Z'$ such that $\phi \circ v = v' \circ f$, and thus a morphism (ψ, f, ϕ) of distinguished triangles.

By Proposition 3.20 of [OT14], the above diagram can be completed to the following diagram:

$$\begin{array}{ccccccc} Y & \xrightarrow{u} & X & \xrightarrow{v} & Z & \xrightarrow{w} & TY \\ \psi \downarrow & & \downarrow f & & \downarrow \phi & & \downarrow T(\psi) \\ Y' & \xrightarrow{\quad} & X' & \xrightarrow{v'} & Z' & \xrightarrow{\quad} & TY' \\ \psi' \downarrow & & \downarrow & & \downarrow & & \downarrow -T(\psi') \\ Y'' & \xrightarrow{\quad} & X'' & \xrightarrow{\quad} & Z'' & \xrightarrow{\quad} & TY'' \\ \psi'' \downarrow & & \downarrow & & \downarrow & & \downarrow -T(\psi'') \\ TY & \xrightarrow{T(u)} & TX & \xrightarrow{T(v)} & TZ & \xrightarrow{-T(w)} & T^2Y \end{array}$$

ac

Then $Y'' \in \mathcal{A}$ and $Z'' \in \mathcal{A}^\perp$ because $Y, Y' \in \mathcal{A}$ and $Z, Z' \in \mathcal{A}^\perp$. Hence $X'' \in \mathcal{D}'$. Therefore \mathcal{D}' is a triangulated subcategory of \mathcal{D} . □

Similarly, we obtain the following.

Proposition 5.9. *Let (\mathcal{D}, T) be a triangulated category and let \mathcal{A} be a strictly full triangulated subcategory of \mathcal{D} . Suppose that $\iota : \mathcal{A} \rightarrow \mathcal{D}$ and ${}^\perp \iota : {}^\perp \mathcal{A} \rightarrow \mathcal{D}$ are the inclusion functors and $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{A}$ is the Verdier localization functor. Then the following conditions are equivalent:*

- (i) \mathcal{A} is left admissible.
- (ii) For each $X \in \mathcal{D}$ there exists a distinguished triangle

$$X^\perp_{\mathcal{A}} \rightarrow X \rightarrow X_{\mathcal{A}} \rightarrow TX^\perp_{\mathcal{A}}$$

with $X^\perp_{\mathcal{A}} \in {}^\perp \mathcal{A}$ and $X_{\mathcal{A}} \in \mathcal{A}$.

- (iii) \mathcal{A} is an epaisse subcategory in \mathcal{D} and the composition functor $Q \circ {}^\perp \iota : {}^\perp \mathcal{A} \rightarrow \mathcal{D}/\mathcal{A}$ is an exact equivalence.
- (iv) \mathcal{A} is an epaisse subcategory in \mathcal{D} and Q admits a left adjoint functor $Q_!$.
- (v) \mathcal{A} is generated by ${}^\perp \mathcal{A}$ and \mathcal{A} as a triangulated category.

In this case, we have a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$, and such a triangle in (ii) is unique up to unique isomorphism. □

Example 5.10. Let \mathcal{A} be an admissible triangulated subcategory of a triangulated category \mathcal{D} . Then any such subcategory \mathcal{A} defines a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with $\mathcal{D}^{\leq 0} = \mathcal{A}$ and $\mathcal{D}^{\geq 0} = (T\mathcal{A})^\perp$ whose heart is trivial. In fact, the converse is true: a t-structure $\mathcal{D}^{\geq 0} \subset \mathcal{D}$ satisfying $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = 0$ is a triangulated subcategory of \mathcal{D} . \square

Corollary 5.11. Let \mathcal{D} be a triangulated category and $\mathcal{A} \subset \mathcal{D}$ a strictly full triangulated subcategory.

- (i) The following two conditions are equivalent:
 - (a) $\iota : \mathcal{A} \rightarrow \mathcal{D}$ admits a right adjoint functor.
 - (b) $\iota^\perp : \mathcal{A}^\perp \rightarrow \mathcal{D}$ admits a left adjoint functor and ${}^\perp(\mathcal{A}^\perp) = \mathcal{A}$.
- (ii) The following two conditions are equivalent:
 - (a) $\iota : \mathcal{A} \rightarrow \mathcal{D}$ admits a left adjoint functor.
 - (b) ${}^\perp \iota : {}^\perp \mathcal{A} \rightarrow \mathcal{D}$ admits a right adjoint functor and $({}^\perp \mathcal{A})^\perp = \mathcal{A}$.

Proof. We give only the proof of (i), since the proof of (ii) is similar to that of (i).

(a) \Rightarrow (b). $\mathcal{A} \subset {}^\perp(\mathcal{A}^\perp)$ is clear. Conversely, for $X \in {}^\perp(\mathcal{A}^\perp)$, by the right admissibility of \mathcal{A} , we have a distinguished triangle $X_{\mathcal{A}} \rightarrow X \rightarrow X_{\mathcal{A}^\perp} \rightarrow TX_{\mathcal{A}}$ with $X_{\mathcal{A}} \in \mathcal{A}$ and $X_{\mathcal{A}^\perp} \in \mathcal{A}^\perp$. By definition, we have $\text{Hom}_{\mathcal{D}}(X, X_{\mathcal{A}^\perp}) = 0$, and then $X_{\mathcal{A}} \simeq X \oplus T^{-1}X_{\mathcal{A}^\perp}$. Hence we see that $X \in \mathcal{A}$ because \mathcal{A} is an epaisse subcategory in \mathcal{D} . In the case, it follows from (i) and (ii) in Proposition 5.8 and Proposition 5.9 that ${}^\perp \iota : {}^\perp \mathcal{A} \rightarrow \mathcal{D}$ admits a right adjoint functor.

(b) \Rightarrow (a). Similarly, this follows from (i) and (ii) in Proposition 5.8 and Proposition 5.9. \square

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