

<研究ノート>

Triangulated Categories III: T-structures

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要旨

この概説は、主に代数幾何学に現れる三角圏や導来圏の性質についてまとめた一連の研究ノートの第3部であり、今回は三角圏の t -構造に関して解説する。 t -構造は三角圏からハートあるいはコアと呼ばれるアーベル部分圏を取り出すことを可能にする。また、第4部以降に扱われる t -構造の貼り合わせや代数多様体上の (代数的) 偏屈層を定義するために必要となる重要な概念である。

第2節において t -構造の定義と基本性質に関して解説する。第3節では t -構造のハートを定義し、それがアーベル部分圏となることを証明する。第4節では三角圏の加法的部分圏が有界 t -構造のハートとなるための必要十分条件を与える。

1 Introduction

This exposition is the third part of our study of triangulated categories in algebraic geometry. We shall explain the definition and fundamental properties of t -structures on triangulated categories and describe their proofs in detail. Main references are [BBD], [GM03], and [KS90].

Through the exposition, we always assume that the translation functor of a triangulated category is an automorphism.

2 Definitions and basic properties

Definition 2.1. Let (\mathcal{D}, T) be a triangulated category. A t -**structure** on \mathcal{D} is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the following conditions:

TS1 $T\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ and $T^{-1}\mathcal{D}^{\geq 0} \subseteq \mathcal{D}^{\geq 0}$.

TS2 $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq 0}$ and $Y \in T^{-1}\mathcal{D}^{\geq 0}$.

TS3 For any $X \in \mathcal{D}$, there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow TA$$

with $A \in \mathcal{D}^{\leq 0}$ and $B \in T^{-1}\mathcal{D}^{\geq 0}$.

The full subcategory $\mathcal{D}^0 := \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is called the **heart** (or **core**) of the t -structure. □

We define $\mathcal{D}^{\leq n} := T^{-n} \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq n} := T^{-n} \mathcal{D}^{\geq 0}$ for each $n \in \mathbb{Z}$.

Remark 2.2. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on a triangulated category (\mathcal{D}, T) with the heart \mathcal{D}^0 . Then the subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ are strictly full for each n . Moreover we have the following properties.

TS1' There exist the following two filtrations of strictly full subcategories of \mathcal{D} :

$$\begin{aligned} \cdots \subset \mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1} \subset \cdots \subset \mathcal{D}^{\leq n} \subset \mathcal{D}^{\leq n+1} \subset \cdots, \\ \cdots \supset \mathcal{D}^{\geq -1} \supset \mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1} \supset \cdots \supset \mathcal{D}^{\geq n} \supset \mathcal{D}^{\geq n+1} \supset \cdots. \end{aligned}$$

TS2' $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq m}$ and $Y \in \mathcal{D}^{\geq n}$ with $m < n$.

TS3' For any $X \in \mathcal{D}$, there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow TA$$

with $A \in \mathcal{D}^{\leq n}$ and $B \in \mathcal{D}^{\geq n+1}$.

Therefore $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$ is a t -structure on \mathcal{D} for each $n \in \mathbb{Z}$ whose heart is $\mathcal{D}^n := T^{-n} \mathcal{D}^0$. □

Lemma 2.3. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on a triangulated category (\mathcal{D}, T) . Then for each $n \in \mathbb{Z}$, the subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ have the zero object.

Proof. For the zero object 0 of \mathcal{D} , we have a distinguished triangle $A \rightarrow 0 \rightarrow B \rightarrow TA$ with $A \in \mathcal{D}^{\leq n}$ and $B \in \mathcal{D}^{\geq n+1}$ by TS3'. Applying the functor $\text{Hom}_{\mathcal{D}}(A, \bullet)$ to the distinguished triangle, we obtain the exact sequence

$$\cdots \rightarrow \text{Hom}(A, T^{-1}B) \rightarrow \text{Hom}(A, A) \rightarrow 0.$$

Since $A \in \mathcal{D}^{\leq n}$ and $T^{-1}B \in \mathcal{D}^{\geq n+2} \subset \mathcal{D}^{\geq n+1}$ by TS1', we see that $\text{Hom}(A, T^{-1}B) = 0$. Hence $\text{Hom}(A, A) = 0$ and $A \simeq 0$. Therefore $0 \in \mathcal{D}^{\leq n}$. Then $0 \simeq B$ and thus $0 \in \mathcal{D}^{\geq n+1}$. □

Definition 2.4. Let \mathcal{D} be an additive category and \mathcal{C} a full subcategory of \mathcal{D} .

1. The **right orthogonal** to \mathcal{C} , denoted by \mathcal{C}^{\perp} , is the strictly full subcategory of \mathcal{D} consisting of objects Y such that for all $X \in \mathcal{C}$, $\text{Hom}_{\mathcal{D}}(X, Y) = 0$.
2. The **left orthogonal** to \mathcal{C} , denoted by ${}^{\perp}\mathcal{C}$, is the strictly full subcategory of \mathcal{D} consisting of objects Y such that for all $X \in \mathcal{C}$, $\text{Hom}_{\mathcal{D}}(Y, X) = 0$. □

Lemma 2.5. Let (\mathcal{D}, T) be a triangulated category and \mathcal{C} a full subcategory of \mathcal{D} . Then for each $i \in \mathbb{Z}$, we have

$$T^i(\mathcal{C}^{\perp}) = (T^i\mathcal{C})^{\perp} \quad \text{and} \quad T^i({}^{\perp}\mathcal{C}) = {}^{\perp}(T^i\mathcal{C}).$$

Proof. For the first assertion, we have

$$\begin{aligned}
 Y \in T^i(\mathcal{C}^\perp) &\iff T^{-i}Y \in \mathcal{C}^\perp \\
 &\iff \text{Hom}_{\mathcal{D}}(X, T^{-i}Y) = 0 \text{ for all } X \in \mathcal{C} \\
 &\iff \text{Hom}_{\mathcal{D}}(T^iX, Y) = 0 \text{ for all } X \in \mathcal{C} \\
 &\iff Y \in (T^i\mathcal{C})^\perp.
 \end{aligned}$$

For the second assertion, we have

$$\begin{aligned}
 X \in T^i(\perp\mathcal{C}) &\iff T^{-i}X \in \perp\mathcal{C} \\
 &\iff \text{Hom}_{\mathcal{D}}(T^{-i}X, Y) = 0 \text{ for all } Y \in \mathcal{C} \\
 &\iff \text{Hom}_{\mathcal{D}}(X, T^iY) = 0 \text{ for all } Y \in \mathcal{C} \\
 &\iff X \in \perp(T^i\mathcal{C}).
 \end{aligned}$$

This completes the proof. \square

Proposition 2.6. *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on a triangulated category (\mathcal{D}, T) .*

Then we have

$$\mathcal{D}^{\geq n} = (\mathcal{D}^{\leq n-1})^\perp \quad \text{and} \quad \mathcal{D}^{\leq n} = \perp(\mathcal{D}^{\geq n+1}).$$

Proof. (1) First we show $\mathcal{D}^{\geq 1} = (\mathcal{D}^{\leq 0})^\perp$. It follows from TS2 that $\mathcal{D}^{\geq 1} \subset (\mathcal{D}^{\leq 0})^\perp$. Conversely, for any object X of $(\mathcal{D}^{\leq 0})^\perp$, we have a distinguished triangle $A \rightarrow X \rightarrow B \rightarrow TA$ with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$ by TS3. Applying the functor $\text{Hom}_{\mathcal{D}}(A, \bullet)$ to the distinguished triangle, we obtain the long exact sequence

$$\cdots \rightarrow \text{Hom}(A, T^{-1}B) \rightarrow \text{Hom}(A, A) \rightarrow \text{Hom}(A, X) \rightarrow \text{Hom}(A, B) \rightarrow \cdots.$$

Then we have $\text{Hom}(A, T^{-1}B) = \text{Hom}(A, X) = 0$ since $TA \in T\mathcal{D}^{\leq 0} = \mathcal{D}^{\leq -1}$ and $B \in \mathcal{D}^{\geq 1}$. Hence $\text{Hom}(A, A) = 0$, and thus $A \simeq 0$. Therefore $X \simeq B \in \mathcal{D}^{\geq 1}$.

(2) By Lemma 2.5, we have

$$\mathcal{D}^{\geq n} = T^{-(n-1)}\mathcal{D}^{\geq 1} = T^{-(n-1)}((\mathcal{D}^{\leq 0})^\perp) = (T^{-(n-1)}\mathcal{D}^{\leq 0})^\perp = (\mathcal{D}^{\leq n-1})^\perp.$$

(1') Similarly, first we show $\mathcal{D}^{\leq 0} = \perp(\mathcal{D}^{\geq 1})$. It follows from TS2 that $\mathcal{D}^{\leq 0} \subset \perp(\mathcal{D}^{\geq 1})$. Conversely, for any object X of $\perp(\mathcal{D}^{\geq 1})$, we have a distinguished triangle $A \rightarrow X \rightarrow B \rightarrow TA$ with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$ by TS3. Applying the functor $\text{Hom}_{\mathcal{D}}(\bullet, B)$ to the distinguished triangle, we obtain the long exact sequence

$$\cdots \rightarrow \text{Hom}(TA, B) \rightarrow \text{Hom}(B, B) \rightarrow \text{Hom}(X, B) \rightarrow \text{Hom}(A, B) \rightarrow \cdots.$$

Then we have $\text{Hom}(TA, B) = \text{Hom}(X, B) = 0$. Hence $\text{Hom}(B, B) = 0$, and thus $B \simeq 0$. Therefore $X \simeq A \in \mathcal{D}^{\leq 0}$.

(2') By Lemma 2.5, we have

$$\mathcal{D}^{\leq n} = T^{-n}\mathcal{D}^{\leq 0} = T^{-n}(\perp(\mathcal{D}^{\geq 1})) = \perp(T^{-n}\mathcal{D}^{\geq 1}) = \perp(\mathcal{D}^{\geq n+1}).$$

This completes the proof. \square

Corollary 2.7. *The following are equivalent:*

- (i) to give a t -structure on a triangulated category (\mathcal{D}, T) ;
- (ii) to give a strictly full subcategory $\mathcal{F} \subset \mathcal{D}$, satisfying $T\mathcal{F} \subset \mathcal{F}$, such that for every object $X \in \mathcal{D}$ there exists a distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow TA$$

in \mathcal{D} with $A \in \mathcal{F}$ and $B \in \mathcal{F}^\perp$;

(iii) to give a strictly full subcategory $\mathcal{G} \subset \mathcal{D}$, satisfying $T^{-1}\mathcal{G} \subset \mathcal{G}$, such that for every object $X \in \mathcal{D}$ there exists a distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow TA$$

in \mathcal{D} with $A \in {}^\perp\mathcal{G}$ and $B \in \mathcal{G}$.

A t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on \mathcal{D} is given by the pair $(\mathcal{F}, (T\mathcal{F})^\perp)$ in the case (ii) and by the pair $({}^\perp\mathcal{G}, T\mathcal{G})$ in the case (iii), respectively. \square

In the rest of this section, (\mathcal{D}, T) is a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t -structure on \mathcal{D} .

Lemma 2.8. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h_i} TX$ ($i = 1, 2$) be two distinguished triangles in \mathcal{D} . If $\text{Hom}_{\mathcal{D}}(TX, Z) = 0$, then $h_1 = h_2$.*

Proof. By the axiom TR4, we have a morphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h_1} & TX \\ \text{id}_X \downarrow & & \downarrow \text{id}_Y & & \downarrow \phi & & \downarrow \text{id}_{TX} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h_2} & TX \end{array}$$

Hence $g = \phi \circ g$ and $h_1 = h_2 \circ \phi$. Since $(\text{id}_Z - \phi) \circ g = 0$, it follows from the long exact sequence arising from the first triangle

$$\text{Hom}(TX, Z) \xrightarrow{h_1^*} \text{Hom}(Z, Z) \xrightarrow{g^*} \text{Hom}(Y, Z) \xrightarrow{f^*} \text{Hom}(Y, Z)$$

that there exists $\psi : TX \rightarrow Z$ such that $\text{id}_Z - \phi = \psi \circ h_1$. By the hypothesis, $\psi = 0$ and thus $\phi = \text{id}_Z$. Therefore $h_1 = h_2$. \square

Proposition 2.9. *We have the following properties.*

(i) *The inclusion functor $\iota_{\leq n} : \mathcal{D}^{\leq n} \rightarrow \mathcal{D}$ admits a right adjoint functor $\tau_{\leq n} : \mathcal{D} \rightarrow \mathcal{D}^{\leq n}$, i.e. there exists a morphism of functors $\eta^n : \iota_{\leq n} \circ \tau_{\leq n} \rightarrow \text{id}_{\mathcal{D}}$ (the **unit** of the adjunction) such that*

$$\text{Hom}_{\mathcal{D}^{\leq n}}(X, \tau_{\leq n} Y) \longrightarrow \text{Hom}_{\mathcal{D}}(\iota_{\leq n} X, Y)$$

is an isomorphism for any $X \in \mathcal{D}^{\leq n}$ and $Y \in \mathcal{D}$.

(ii) *The inclusion functor $\iota_{\geq n} : \mathcal{D}^{\geq n} \rightarrow \mathcal{D}$ admits a left adjoint functor $\tau_{\geq n} : \mathcal{D} \rightarrow \mathcal{D}^{\geq n}$, i.e. there exists a morphism of functors $\varepsilon^n : \text{id}_{\mathcal{D}} \rightarrow \iota_{\geq n} \circ \tau_{\geq n}$ (the **counit** of the adjunction) such that*

$$\text{Hom}_{\mathcal{D}^{\geq n}}(\tau_{\geq n} X, Y) \longrightarrow \text{Hom}_{\mathcal{D}}(X, \iota_{\geq n} Y)$$

is an isomorphism for any $X \in \mathcal{D}$ and $Y \in \mathcal{D}^{\geq n}$.

(iii) *There exists a unique morphism $d^{n+1}(X) : \tau_{\geq n+1} X \rightarrow T\tau_{\leq n} X$ such that*

$$(2.1) \quad \tau_{\leq n} X \xrightarrow{\eta^n(X)} X \xrightarrow{\varepsilon^{n+1}(X)} \tau_{\geq n+1} X \xrightarrow{d^{n+1}(X)} T\tau_{\leq n} X$$

is a distinguished triangle. Moreover, $d^{n+1} : \tau^{\geq n+1} \rightarrow T \circ \tau_{\leq n}$ is a morphism of functors. The triangle (2.1) is a unique (up to isomorphisms) distinguished triangle A

$$\rightarrow X \rightarrow B \rightarrow TA \text{ with } A \in \mathcal{D}^{\leq n} \text{ and } B \in \mathcal{D}^{\geq n+1}.$$

The functors $\tau_{\leq n}$ and $\tau_{\geq n}$ are called the **truncation functors** with respect to the t -structure.

Proof. (i) We may assume that the t -structure is $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ by Remark 2.2. In this case, $n = 0$. By TS3, for $Y \in \mathcal{D}$, there exists a distinguished triangle

$$(2.2) \quad Y_0 \xrightarrow{f} Y \xrightarrow{g} Y_1 \xrightarrow{h} TY_0$$

with $Y_0 \in \mathcal{D}^{\leq 0}$ and $Y_1 \in \mathcal{D}^{\geq 1}$. We see that $\text{Hom}_{\mathcal{D}}(X, Y_0) \xrightarrow{f^*} \text{Hom}_{\mathcal{D}}(X, Y)$ is an isomorphism for any $X \in \mathcal{D}^{\leq 0}$. Indeed, this follows from the long exact sequence

$$\text{Hom}(X, T^{-1}Y_1) \xrightarrow{(-T^{-1}(h))^*} \text{Hom}(X, Y_0) \xrightarrow{f^*} \text{Hom}(X, Y) \xrightarrow{g^*} \text{Hom}(X, Y_1),$$

$T^{-1}Y_1 \in \mathcal{D}^{\geq 2} \subset \mathcal{D}^{\geq 1}$ and TS2. Then we set $\tau_{\leq 0}(Y) := Y_0$ (up to isomorphisms).

Let $\phi : Y \rightarrow Y'$ be any morphism in \mathcal{D} . Then there exists an another distinguished triangle

$$(2.3) \quad Y_0' \xrightarrow{f'} Y' \xrightarrow{g'} Y_1' \xrightarrow{h'} TY_0'$$

with $Y_0' \in \mathcal{D}^{\leq 0}$ and $Y_1' \in \mathcal{D}^{\geq 1}$. We show that there exists a unique morphism $\phi_0 : Y_0 \rightarrow Y_0'$ making the following diagram commutative.

$$\begin{array}{ccccccc} Y_0 & \xrightarrow{f} & Y & \longrightarrow & Y_1 & \longrightarrow & TY_0 \\ & & \downarrow \phi & & & & \\ \phi_0 \downarrow \text{dotted} & & & & & & \\ Y_0' & \longrightarrow & Y' & \longrightarrow & Y_1' & \longrightarrow & TY_0' \end{array}$$

Applying the functor $\text{Hom}_{\mathcal{D}}(Y_0, \bullet)$ to the second triangle, $\text{Hom}_{\mathcal{D}}(Y_0, T^{-1}Y_1') = \text{Hom}_{\mathcal{D}}(Y_0, Y_1') = 0$ implies an isomorphism $\text{Hom}_{\mathcal{D}}(Y_0, Y_0') \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(Y_0, Y')$. In that case, ϕ_0 is the preimage of $\phi \circ f$ under this isomorphism. Then we set $\tau_{\leq 0}(\phi) := \phi_0$. This proves that there exists a morphism of functors $\iota_{\leq 0} \circ \tau_{\leq 0} \rightarrow \text{id}_{\mathcal{D}}$. Therefore we obtain a right adjoint functor $\tau_{\leq 0} : \mathcal{D} \rightarrow \mathcal{D}^{\leq 0}$.

(ii) Similarly, we assume that the t -structure is $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. In this case, $n = 1$. For $Y \in \mathcal{D}$, it follows from the distinguished triangle (2.2) that $\text{Hom}_{\mathcal{D}}(Y_1, X) \xrightarrow{g^*} \text{Hom}_{\mathcal{D}}(Y, X)$ is an isomorphism for any $X \in \mathcal{D}^{\geq 1}$. Indeed, this follows from the long exact sequence:

$$\text{Hom}(TY_0, X) \xrightarrow{h^*} \text{Hom}(Y_1, X) \xrightarrow{g^*} \text{Hom}(Y, X) \xrightarrow{f^*} \text{Hom}(Y_0, X)$$

and $TY_0 \in \mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$. Then we set $\tau_{\geq 1}(Y) := Y_1$ (up to isomorphisms).

Let $\phi : Y \rightarrow Y'$ be any morphism in \mathcal{D} . Then we show that for the another distinguished triangle (2.3), there exists a unique morphism $\phi_1 : Y_1 \rightarrow Y_1'$ making the following diagram commutative.

$$\begin{array}{ccccccc} Y_0 & \longrightarrow & Y & \longrightarrow & Y_1 & \longrightarrow & TY_0 \\ & & \downarrow \phi & & \downarrow \text{dotted } \phi_1 & & \\ Y_0' & \longrightarrow & Y' & \xrightarrow{g'} & Y_1' & \longrightarrow & TY_0' \end{array}$$

Applying the functor $\text{Hom}_{\mathcal{D}}(\bullet, Y_1')$ to the first triangle, $\text{Hom}_{\mathcal{D}}(TY_0, Y_1') = \text{Hom}_{\mathcal{D}}(Y_0, Y_1') = 0$ implies an isomorphism $\text{Hom}_{\mathcal{D}}(Y_1, Y_1') \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(Y, Y_1')$. In that case, ϕ_1 is

the preimage of $g' \circ \phi$ under this isomorphism. Then we set $\tau_{\geq 1}(\phi) := \phi_1$. This proves that there exists a morphism of functors $\text{id}_{\mathcal{D}} \rightarrow \iota_{\geq 1} \circ \tau_{\geq 1}$. Therefore we obtain a left adjoint functor $\tau_{\geq 1} : \mathcal{D} \rightarrow \mathcal{D}^{\geq 1}$.

(iii) The first part follows from TS3, Lemma 2.8 and TS2'. We show that d is a morphism of functors. For $\phi : X \rightarrow Y$, there exists by TR4 a morphism of distinguished triangles:

$$\begin{array}{ccccccc} \tau_{\leq n} X & \longrightarrow & X & \longrightarrow & \tau_{\geq n+1} X & \xrightarrow{d^{n+1}(X)} & T\tau_{\leq n} X \\ \tau_{\leq n}(\phi) \downarrow & & \downarrow \phi & & \downarrow \psi & & \downarrow T(\tau_{\leq n}(\phi)) \\ \tau_{\leq n} Y & \longrightarrow & Y & \longrightarrow & \tau_{\geq n+1} Y & \xrightarrow{d^{n+1}(Y)} & T\tau_{\leq n} Y. \end{array}$$

It follows from Proposition 3.9 of [OT14b] that $\psi = \tau_{\geq n+1}(\phi)$. □

Proposition 2.10. *We have the following two isomorphisms of functors:*

$$\tau_{\leq n} \circ T^m \simeq T^m \circ \tau_{\leq n+m} \quad \text{and} \quad \tau_{\geq n} \circ T^m \simeq T^m \circ \tau_{\geq n+m}.$$

Proof. By the adjunction, the first isomorphism follows from

$$\begin{aligned} \text{Hom}_{\mathcal{D}^{\leq n}}(Y, \tau_{\leq n} T^m X) &\simeq \text{Hom}_{\mathcal{D}}(\iota_{\leq n} Y, T^m X) \\ &\simeq \text{Hom}_{\mathcal{D}}(T^{-m} \iota_{\leq n} Y, X) \\ &\simeq \text{Hom}_{\mathcal{D}}(\iota_{\leq n+m} T^{-m} Y, X) \\ &\simeq \text{Hom}_{\mathcal{D}^{\leq n+m}}(T^{-m} Y, \tau_{\leq n+m} X) \\ &\simeq \text{Hom}_{\mathcal{D}^{\leq n}}(Y, T^m \tau_{\leq n+m} X) \end{aligned}$$

for any $Y \in \mathcal{D}^{\leq n}$ and any $X \in \mathcal{D}$.

Similarly, the second isomorphism follows from

$$\begin{aligned} \text{Hom}_{\mathcal{D}^{\geq n}}(\tau_{\geq n} T^m X, Y) &\simeq \text{Hom}_{\mathcal{D}}(T^m X, \iota_{\geq n} Y) \\ &\simeq \text{Hom}_{\mathcal{D}}(X, T^{-m} \iota_{\geq n} Y) \\ &\simeq \text{Hom}_{\mathcal{D}}(X, \iota_{\geq n+m} T^{-m} Y) \\ &\simeq \text{Hom}_{\mathcal{D}^{\geq n+m}}(\tau_{\geq n+m} X, T^{-m} Y) \\ &\simeq \text{Hom}_{\mathcal{D}^{\geq n}}(T^m \tau_{\geq n+m} X, Y) \end{aligned}$$

for any $Y \in \mathcal{D}^{\geq n}$ and any $X \in \mathcal{D}$. □

We sometimes write $\tau_{>n}$ and $\tau_{<n}$ instead of $\tau_{\geq n+1}$ and $\tau_{\leq n-1}$, respectively, and similarly for $\mathcal{D}^{<n}$ and $\mathcal{D}^{>n}$.

Proposition 2.11. *Let X be an object of \mathcal{D} .*

(i) *The following three conditions are equivalent:*

- (a) $X \in \mathcal{D}^{\leq n}$.
- (b) $\tau_{>n} X = 0$.
- (c) *The unit $\eta^n(X) : \tau_{\leq n} X \rightarrow X$ of the adjunction is an isomorphism.*

(ii) *The following three conditions are equivalent:*

- (a) $X \in \mathcal{D}^{\geq n}$.

(b) $\tau_{<n} X = 0$.

(c) The counit $\varepsilon^n(X) : X \rightarrow \tau_{\geq n} X$ of the adjunction is an isomorphism.

Proof. (i) follows from the distinguished triangle (2.1). We obtain the exact sequence

$$\mathrm{Hom}(T \tau_{\leq n} X, \tau_{\geq n+1} X) \rightarrow \mathrm{Hom}(\tau_{\geq n+1} X, \tau_{\geq n+1} X) \rightarrow \mathrm{Hom}(X, \tau_{\geq n+1} X).$$

Then by TS2', we see that the both edges are zero, and thus the middle is zero. Hence $\tau_{>n} X \simeq 0$. Similarly, we obtain (ii). \square

Proposition 2.12. *Let $X' \rightarrow X \rightarrow X'' \rightarrow TX'$ be a distinguished triangle in \mathcal{D} .*

- (i) (a) *If $X', X'' \in \mathcal{D}^{\leq n}$, then $X \in \mathcal{D}^{\leq n}$.*
- (b) *If $X', X \in \mathcal{D}^{\leq n}$, then $X'' \in \mathcal{D}^{\leq n}$.*
- (c) *If $X, X'' \in \mathcal{D}^{\leq n}$, then $X' \in \mathcal{D}^{\leq n+1}$.*
- (ii) (a) *If $X', X'' \in \mathcal{D}^{\geq n}$, then $X \in \mathcal{D}^{\geq n}$.*
- (b) *If $X', X \in \mathcal{D}^{\geq n}$, then $X'' \in \mathcal{D}^{\geq n-1}$.*
- (c) *If $X, X'' \in \mathcal{D}^{\geq n}$, then $X' \in \mathcal{D}^{\geq n}$.*

In particular, $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ are extension-closed for each n .

Proof. (i) (a) From the distinguished triangle, we have the exact sequence

$$\mathrm{Hom}(X'', \tau_{>n} X) \rightarrow \mathrm{Hom}(X, \tau_{>n} X) \rightarrow \mathrm{Hom}(X', \tau_{>n} X).$$

Then since the both edges are zero, and thus the middle is zero. Moreover, from the distinguished triangle (2.1), we obtain the exact sequence

$$\mathrm{Hom}(T \tau_{\leq n} X, \tau_{>n} X) \rightarrow \mathrm{Hom}(\tau_{>n} X, \tau_{>n} X) \rightarrow \mathrm{Hom}(X, \tau_{>n} X)$$

and thus $\mathrm{Hom}(\tau_{>n} X, \tau_{>n} X) = 0$. Hence $\tau_{>n} X \simeq 0$. By Proposition 2.11.(i), we obtain $X \in \mathcal{D}^{\leq n}$.

(b) Since $TX' \in \mathcal{D}^{\leq n-1} \subset \mathcal{D}^{\leq n}$, this follows from a distinguished triangle $X \rightarrow X'' \rightarrow TX' \rightarrow TX$ and (a).

(c) Since $X \in \mathcal{D}^{\leq n} \subset \mathcal{D}^{\leq n+1}$, this follows from a distinguished triangle $T^{-1}X'' \rightarrow X' \rightarrow X \rightarrow X''$ and (a).

(ii) (a) From the distinguished triangle, we have the exact sequence

$$\mathrm{Hom}(\tau_{<n} X, X') \rightarrow \mathrm{Hom}(\tau_{<n} X, X) \rightarrow \mathrm{Hom}(\tau_{<n} X, X'').$$

Then since the both edges are zero, so is the middle. Moreover, from the distinguished triangle (2.1), we obtain the exact sequence

$$\mathrm{Hom}(\tau_{<n} X, T^{-1} \tau_{\geq n} X) \rightarrow \mathrm{Hom}(\tau_{<n} X, \tau_{<n} X) \rightarrow \mathrm{Hom}(\tau_{<n} X, X)$$

and thus $\mathrm{Hom}(\tau_{<n} X, \tau_{<n} X) = 0$. Hence $\tau_{<n} X \simeq 0$. By Proposition 2.11.(ii), we obtain $X \in \mathcal{D}^{\geq n}$.

(b) Since $X \in \mathcal{D}^{\geq n} \subset \mathcal{D}^{\geq n-1}$, this follows from a distinguished triangle $X \rightarrow X'' \rightarrow TX' \rightarrow TX$ and (a).

(c) Since $T^{-1} X'' \in \mathcal{D}^{\geq n+1} \subset \mathcal{D}^{\geq n}$, this follows from a distinguished triangle $T^{-1} X'' \rightarrow X' \rightarrow X \rightarrow X''$ and (a). \square

Proposition 2.13. *The strictly full subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ of \mathcal{D} are additive categories for each $n \in \mathbb{Z}$. Moreover they are closed under direct summands.*

Proof. Since the triangulated category \mathcal{D} is additive, the full subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ are pre-additive for each n . In Lemma 2.3, we proved that for each n , the subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ have the zero object 0 . For any objects $X', X'' \in \mathcal{D}$, we have a distinguished triangle $X' \rightarrow X' \oplus X'' \rightarrow X'' \rightarrow TX'$. Hence by Proposition 2.12, if X' and X'' belong to $\mathcal{D}^{\geq n}$, then $X' \oplus X''$ also belongs to $\mathcal{D}^{\geq n}$. Similarly, if X' and X'' belong to $\mathcal{D}^{\leq 0}$, then $X' \oplus X''$ also belongs to $\mathcal{D}^{\leq n}$. Therefore $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ are additive categories.

Let us show that $\mathcal{D}^{\leq n}$ is closed under direct summands. Let X, Y be any objects in \mathcal{D} , and assume that their direct sum $X \oplus Y$ belongs to $\mathcal{D}^{\leq n}$. Then we have two distinguished triangles $X' \rightarrow X \rightarrow X'' \rightarrow TX'$ and $Y' \rightarrow Y \rightarrow Y'' \rightarrow TY'$ with $X', Y' \in \mathcal{D}^{\leq n}$ and $X'', Y'' \in \mathcal{D}^{\geq n+1}$.

Taking a direct sum of the triangles, by Proposition 2.12, we obtain $X'' \oplus Y'' \in \mathcal{D}^{\leq n}$ since $X' \oplus Y'$ and $X \oplus Y$ belong to $\mathcal{D}^{\leq n}$. Hence we get $X'' \oplus Y'' = 0$ and thus $X'' = Y'' = 0$, because $\mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq n+1} = \{0\}$. Therefore $X \simeq X'$ and $Y \simeq Y'$, and thus X and Y belong to $\mathcal{D}^{\leq n}$.

Similarly, let us show that $\mathcal{D}^{\leq n}$ is closed under direct summands. Let X, Y be any objects in \mathcal{D} , and assume their direct sum $X \oplus Y$ belongs to $\mathcal{D}^{\geq n}$. Then we have two distinguished triangles $X' \rightarrow X \rightarrow X'' \rightarrow TX'$ and $Y' \rightarrow Y \rightarrow Y'' \rightarrow TY'$ with $X', Y' \in \mathcal{D}^{\leq n-1}$ and $X'', Y'' \in \mathcal{D}^{\geq n}$. Taking a direct sum of the triangles, by Proposition 2.12, we obtain $X' \oplus Y' \in \mathcal{D}^{\geq n}$ since $X \oplus Y$ and $X'' \oplus Y''$ belong to $\mathcal{D}^{\geq n}$. Hence we get $X' \oplus Y' = 0$ and thus $X' = Y' = 0$, because $\mathcal{D}^{\leq n-1} \cap \mathcal{D}^{\geq n} = \{0\}$. Therefore $X \simeq X''$ and $Y \simeq Y''$, and thus X and Y belong to $\mathcal{D}^{\geq n}$.

Proposition 2.14. *Let a and b be two integers.*

(i) *If $a \leq b$, then we have*

$$\tau_{\geq b} \circ \tau_{\geq a} \simeq \tau_{\geq a} \circ \tau_{\geq b} \simeq \tau_{\geq b} \quad \text{and} \quad \tau_{\leq b} \circ \tau_{\leq a} \simeq \tau_{\leq a} \circ \tau_{\leq b} \simeq \tau_{\leq a}.$$

(ii) *If $a > b$, then we have*

$$\tau_{\leq b} \circ \tau_{\geq a} = \tau_{\geq a} \circ \tau_{\leq b} = 0.$$

(iii) *We have*

$$\tau_{\geq a} \circ \tau_{\leq b} \simeq \tau_{\leq b} \circ \tau_{\geq a}.$$

More precisely, for $X \in \mathcal{D}$, there exists a unique morphism

$$\Phi^{[a,b]}(X) : \tau_{\geq a} \tau_{\leq b} X \rightarrow \tau_{\leq b} \tau_{\geq a} X$$

such that the diagram:

$$\begin{array}{ccc} \tau_{\leq b} X & \xrightarrow{\eta^b(X)} & X & \xrightarrow{\varepsilon^a(X)} & \tau_{\geq a} X \\ \varepsilon^a(\tau_{\leq b} X) \downarrow & & & & \uparrow \eta^b(\tau_{\geq a} X) \\ \tau_{\geq a} \tau_{\leq b} X & \xrightarrow{\Phi^{[a,b]}(X)} & & & \tau_{\leq b} \tau_{\geq a} X \end{array}$$

commutes, and moreover $\Phi^{[a,b]}$ is an isomorphism.

Proof. (i) $\tau_{\geq a} \circ \tau_{\geq b} \simeq \tau_{\geq b}$ follows from Proposition 2.11.(ii). For any $X \in \mathcal{D}$ and $Y \in \mathcal{D}^{\geq b}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^{\geq b}}(\tau_{\geq b} \tau_{\geq a} X, Y) &\simeq \mathrm{Hom}_{\mathcal{D}^{\geq a}}(\tau_{\geq a} X, Y) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X, Y) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^{\geq b}}(\tau_{\geq b} X, Y). \end{aligned}$$

Similarly, we obtain other isomorphisms.

(ii) This follows from Proposition 2.11.

(iii) By (ii), we may assume $a \leq b$. By (i), there exist the following two distinguished triangles:

$$(2.4) \quad \tau_{\leq b} \tau_{\geq a} X \xrightarrow{\eta^b(\tau_{\geq a} X)} \tau_{\geq a} X \xrightarrow{\varepsilon^{b+1}(\tau_{\geq a} X)} \tau_{> b} X \longrightarrow T \tau_{\leq b} \tau_{\geq a} X,$$

$$\tau_{< a} X \xrightarrow{\eta^{a-1}(\tau_{\leq b} X)} \tau_{\leq b} X \xrightarrow{\varepsilon^a(\tau_{\leq b} X)} \tau_{\geq a} \tau_{\leq b} X \longrightarrow T \tau_{< a} X.$$

By Proposition 2.12, we have $T^a \tau_{\leq b} \tau_{\geq a} X \in \mathcal{D}^{\geq 0}$ and $T^b \tau_{\geq a} \tau_{\leq b} X \in \mathcal{D}^{\leq 0}$. Then $\tau_{\leq b} \tau_{\geq a} X \in \mathcal{D}^{\geq a}$ and $\tau_{\geq a} \tau_{\leq b} X \in \mathcal{D}^{\leq b}$. Hence $\tau_{\leq b} \tau_{\geq a} X$ and $\tau_{\geq a} \tau_{\leq b} X$ belong to $\mathcal{D}^{\geq a} \cap \mathcal{D}^{\leq b}$. Therefore we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^{\geq a} \cap \mathcal{D}^{\leq b}}(\tau_{\geq a} \tau_{\leq b} X, \tau_{\leq b} \tau_{\geq a} X) &\simeq \mathrm{Hom}_{\mathcal{D}^{\geq a}}(\tau_{\geq a} \tau_{\leq b} X, \tau_{\geq a} X) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq b} X, \tau_{\geq a} X). \end{aligned}$$

This gives the existence and the uniqueness of ϕ .

Now we shall show that ϕ is an isomorphism. We apply the octahedral axiom TR5 to the morphisms $\tau_{< a} X \xrightarrow{\eta^{a-1}(\tau_{\leq b} X)} \tau_{\leq b} X \xrightarrow{\eta^b(X)} X$:

$$\begin{array}{ccccccc} \tau_{< a} X & \xrightarrow{\eta^{a-1}(\tau_{\leq b} X)} & \tau_{\leq b} X & \xrightarrow{\varepsilon^a(\tau_{\leq b} X)} & \tau_{\geq a} \tau_{\leq b} X & \longrightarrow & T \tau_{< a} X \\ \mathrm{id} \downarrow & & \downarrow \eta^b(X) & & \downarrow u & & \downarrow T(\mathrm{id}) \\ \tau_{< a} X & \xrightarrow{\eta^b(X) \circ \eta^{a-1}(\tau_{\leq b} X)} & X & \xrightarrow{l} & \tau_{\geq a} X & \longrightarrow & T \tau_{< a} X \\ \eta^{a-1}(\tau_{\leq b} X) \downarrow & & \downarrow \mathrm{id} & & \downarrow v & & \downarrow T(f) \\ \tau_{\leq b} X & \xrightarrow{\eta^b(X)} & X & \xrightarrow{\varepsilon^{b+1}(X)} & \tau_{> b} X & \longrightarrow & T \tau_{\leq b} X \\ \varepsilon^a(\tau_{\leq b} X) \downarrow & & \downarrow l & & \downarrow \mathrm{id} & & \downarrow \\ \tau_{\geq a} \tau_{\leq b} X & \xrightarrow{u} & \tau_{\geq a} X & \xrightarrow{v} & \tau_{> b} X & \xrightarrow{w} & T \tau_{\geq a} \tau_{\leq b} X \end{array}$$

Moreover, by TS2', we have

$$\mathrm{Hom}(\tau_{\geq a} \tau_{\leq b} X, \tau_{> b} X) = 0 \text{ and } \mathrm{Hom}(\tau_{\geq a} \tau_{\leq b} X, T^{-1} \tau_{> b} X) = 0.$$

Hence, by Proposition 3.20 of [OT14b], there exists a unique morphism of distinguished triangles:

$$\begin{array}{ccccccc} \tau_{\geq a} \tau_{\leq b} X & \xrightarrow{u} & \tau_{\geq a} X & \xrightarrow{v} & \tau_{> b} X & \xrightarrow{w} & T \tau_{\geq a} \tau_{\leq b} X \\ \alpha \downarrow & & \parallel & & \downarrow \beta & & \downarrow T(\alpha) \\ \tau_{\leq b} \tau_{\geq a} X & \xrightarrow{\eta^b(\tau_{\geq a} X)} & \tau_{\geq a} X & \xrightarrow{\varepsilon^{b+1}(\tau_{\geq a} X)} & \tau_{> b} X & \longrightarrow & T \tau_{\leq b} \tau_{\geq a} X. \end{array}$$

Therefore we obtain $\alpha = \Phi^{[a,b]}$ is an isomorphism and β is the identity. \square

3 The hearts of t -structures

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on a triangulated category (\mathcal{D}, T) and let $\mathcal{D}^0 = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ be its heart.

Definition 3.1. We define the functor $H_{\mathcal{D}}^0 : \mathcal{D} \rightarrow \mathcal{D}^0$ by $H_{\mathcal{D}}^0(X) := \tau_{\geq 0} \tau_{\leq 0} X (\simeq \tau_{\leq 0} \tau_{\geq 0} X)$ for $X \in \mathcal{D}$. We also set $H_{\mathcal{D}}^n(X) := H^0(T^n X) (= T^n \tau_{\geq n} \tau_{\leq n} X \simeq T^n \tau_{\leq n} \tau_{\geq n} X)$ for each $n \in \mathbb{Z}$. □

We often write H^n instead of $H_{\mathcal{D}}^n$.

Theorem 3.2. *The heart $\mathcal{D}^0 = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an abelian category.*

Proof. (1) \mathcal{D}^0 is an additive category: This follows from Proposition 2.13.

(2) Kernel and Cokernel in \mathcal{D}^0 : Let $f: X \rightarrow Y$ be a morphism in \mathcal{D}^0 , and let us embed f into a distinguished triangle

$$(3.1) \quad X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX.$$

Then $Z \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$ by Proposition 2.12. We shall prove that

$$(3.2) \quad \text{Coker}(f) \simeq \tau_{\geq 0} Z \quad \text{and} \quad \text{Ker}(f) \simeq \tau_{\leq 0} T^{-1} Z.$$

Note that $\tau_{\geq 0} Z \simeq \tau_{\geq 0} \tau_{\leq 0} Z \simeq H^0(Z)$ since $Z \in \mathcal{D}^{\leq 0}$ and

$$\tau_{\leq 0} T^{-1} Z \simeq \tau_{\leq 0} T^{-1} \tau_{\geq -1} Z \simeq \tau_{\leq 0} \tau_{\geq 0} T^{-1} Z \simeq H^0(T^{-1} Z)$$

since $Z \in \mathcal{D}^{\geq -1}$.

In order to prove (3.2), take any object $W \in \mathcal{D}^0$ and consider the long exact sequences

$$(3.3) \quad \text{Hom}(TX, W) \xrightarrow{h^*} \text{Hom}(Z, W) \xrightarrow{g^*} \text{Hom}(Y, W) \xrightarrow{f^*} \text{Hom}(X, W)$$

and

$$(3.4) \quad \text{Hom}(W, T^{-1} Y) \xrightarrow{(-T^{-1}(g))_*} \text{Hom}(W, T^{-1} Z) \xrightarrow{(-T^{-1}(h))_*} \text{Hom}(W, X) \xrightarrow{f_*} \text{Hom}(W, Y).$$

Then, since $W \in \mathcal{D}^0$, we have $\text{Hom}(TX, W) = \text{Hom}(W, T^{-1} Y) = 0$ by TS2', and we obtain the morphisms

$$\text{Hom}(Z, W) \xleftarrow{\varepsilon^0(Z)^*} \text{Hom}(\tau_{\geq 0} Z, W)$$

and

$$\text{Hom}(W, T^{-1} Z) \xleftarrow{\eta^0(T^{-1} Z)_*} \text{Hom}(W, \tau_{\leq 0} T^{-1} Z)$$

by the adjunction isomorphisms. Hence the long exact sequences (3.3) and (3.4) yield exact sequences

$$0 \longrightarrow \text{Hom}(\tau_{\geq 0} Z, W) \xrightarrow{g^* \circ \varepsilon^0(Z)^*} \text{Hom}(Y, W) \xrightarrow{f^*} \text{Hom}(X, W)$$

and

$$0 \longrightarrow \text{Hom}(W, \tau_{\leq 0} T^{-1} Z) \xrightarrow{(-T^{-1}(h))_* \circ \eta^0(T^{-1} Z)_*} \text{Hom}(W, X) \xrightarrow{f_*} \text{Hom}(W, Y).$$

Then since

$$g^* \circ \varepsilon^0(Z)^* = (\varepsilon^0(Z) \circ g)^*$$

and

$$(-T^{-1}(h))_* \circ \eta^0(T^{-1} Z)_* = ((-T^{-1} h) \circ \eta^0(T^{-1} Z))^*,$$

we have the following commutative diagrams

$$(3.5) \quad \begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\pi} & \text{Coker}(f) & \longrightarrow & 0 \\ & & \parallel & & \downarrow \simeq & & \\ & & Y & \xrightarrow{\varepsilon^0(Z) \circ g} & \tau_{\geq 0} Z & \longrightarrow & 0 \end{array}$$

and

$$(3.6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \tau_{\leq 0} T^{-1} Z & \xrightarrow{(-T^{-1}(h)) \circ \eta^0(T^{-1} Z)} & X & & \\ & & \downarrow \simeq & & \parallel & & \\ 0 & \longrightarrow & \text{Ker}(f) & \xrightarrow{l} & X & \xrightarrow{f} & Y. \end{array}$$

Therefore we obtain the isomorphisms (3.2).

(3) Coimage and Image in \mathcal{D}^0 : Consider the distinguished triangles

$$Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY$$

induced by (3.1) and

$$Z \xrightarrow{\varepsilon^0(Z)} \tau_{\geq 0} Z \xrightarrow{d^0(Z)} T\tau_{\leq -1} Z \xrightarrow{-T\eta^{-1}(Z)} TZ.$$

We embed the morphism $\varepsilon^0(Z) \circ g : Y \rightarrow \tau_{\geq 0} Z$ into a distinguished triangle

$$(3.7) \quad I \longrightarrow Y \xrightarrow{\varepsilon^0(Z) \circ g} \tau_{\geq 0} Z \longrightarrow TI.$$

By Proposition 2.12, we see that I belongs to $\mathcal{D}^{\geq 0}$.

We apply the octahedral axiom TR5 to $Y \xrightarrow{g} Z \xrightarrow{\varepsilon^0(Z)} \tau_{\geq 0} Z$, and then we have

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX & \xrightarrow{-Tf} & TY \\ \parallel & & \downarrow \varepsilon^0(Z) & & \vdots u & & \parallel \\ Y & \xrightarrow{\varepsilon^0(Z) \circ g} & \tau_{\geq 0} Z & \xrightarrow{l} & TI & \longrightarrow & TY \\ \downarrow g & & \parallel & & \vdots v & & \downarrow \\ Z & \xrightarrow{\varepsilon^0(Z)} & \tau_{\geq 0} Z & \xrightarrow{d^0(Z)} & T\tau_{\leq -1} Z & \xrightarrow{-T\eta^{-1}(Z)} & TZ \\ \downarrow h & & \downarrow l & & \parallel & & \downarrow Th \\ TX & \xrightarrow{\dots u} & TI & \xrightarrow{\dots v} & T\tau_{\leq -1} Z & \xrightarrow{\dots w} & T^2 X. \end{array}$$

Then we obtain the distinguished triangle

$$T^{-1} \tau_{\leq -1} Z \xrightarrow{T^{-2} w} X \longrightarrow I \longrightarrow \tau_{< 0} Z,$$

where we set $T^{-2} w = T^{-1} h \circ (-T^{-1}(\eta^{-1} Z))$.

Note that $T^{-1}(\eta^{-1} Z) = \eta^0(T^{-1} Z)$. Indeed, we have, by the adjunction morphisms and Proposition 2.10,

$$\text{Hom}(T^{-1} \tau_{\leq -1} Z, T^{-1} Z) \simeq \text{Hom}(\tau_{\leq 0} T^{-1} Z, T^{-1} Z).$$

Hence $I \in \mathcal{D}^{\leq 0}$, and thus $I \in \mathcal{D}^0$, i.e. $I \simeq \tau_{\geq 0} I \simeq \tau_{\leq 0} I$. Since $T^{-1} \tau_{\leq -1} Z \simeq \tau_{\leq 0} T^{-1} Z$,

it follows from the diagrams (3.5) and (3.6) that

$$\mathrm{Coim}(f) = \mathrm{Coker}(t) \simeq \mathrm{Coker}((-T^{-1}h) \circ \eta^0(T^{-1}Z)) \simeq \tau_{\geq 0} I \simeq I$$

and

$$\mathrm{Im}(f) = \mathrm{Ker}(\pi) \simeq \mathrm{Ker}(\varepsilon^0(Z) \circ g) \simeq \tau_{\leq 0} T^{-1} T I \simeq \tau_{\leq 0} I \simeq I.$$

Therefore we obtain $\mathrm{Coim}(f) \simeq \mathrm{Im}(f)$. This completes the proof. \square

Proposition 3.3. *If $X' \rightarrow X \rightarrow X'' \rightarrow TX'$ is a distinguished triangle in \mathcal{D} and if X' and X'' belong to \mathcal{D}^0 , then X belongs to \mathcal{D}^0 . Hence the heart \mathcal{D}^0 is extension-closed.*

Proof. This follows from Proposition 2.12 \square

Proposition 3.4. Let $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ be a short exact sequence in \mathcal{D}^0 .

(i) *There exists a unique morphism $h : Z \rightarrow TX$ in \mathcal{D} such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a distinguished triangle in \mathcal{D} .*

(ii) *Conversely, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a distinguished triangle in \mathcal{D} with $X, Z \in \mathcal{D}^0$, then $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is a short exact sequence in \mathcal{D}^0 .*

Proof. First we embed f into a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g'} W \xrightarrow{h'} TX$. By (3.2), we have

$$Z \simeq \mathrm{Coker}(f) \simeq \tau_{\geq 0} W \quad \text{and} \quad 0 \simeq \mathrm{Ker}(f) \simeq \tau_{\leq 0} T^{-1} W.$$

Then it follows from Proposition 2.11.(ii) that $W \in \mathcal{D}^{\geq 0}$ and $\varepsilon^0(W) : W \xrightarrow{\sim} \tau_{\leq 0} W$ is an isomorphism. Moreover by the diagram (3.5), $\varepsilon^0(W) \circ g' = g$. Hence we obtain a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ in \mathcal{D} . The uniqueness of h follows from Lemma 2.8.

Conversely, assume that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a distinguished triangle in \mathcal{D} with $X, Z \in \mathcal{D}^0$. By Proposition 3.3, Y also belongs to \mathcal{D}^0 . Then

$$\mathrm{Coker}(f) \simeq \tau_{\geq 0} Z \simeq Z \quad \text{and} \quad \mathrm{Ker}(f) \simeq \tau_{\leq 0} T^{-1} Z \simeq 0$$

imply $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is a short exact sequence in \mathcal{D}^0 . \square

Proposition 3.5. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on a triangulated category (\mathcal{D}, T) and \mathcal{D}^0 the heart of the t -structure. Let $n \in \mathbb{Z}$ be an integer. Then, for $X \in \mathcal{D}^{\leq n}$ and $Y \in \mathcal{D}^{\geq n}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{D}^0}(H^n(X), H^n(Y)).$$

Proof. This follows from Propositions 2.9 and 2.11. \square

Proposition 3.6. *The functor $H^0 : \mathcal{D} \rightarrow \mathcal{D}^0$ is a cohomological functor.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow TX$ be a distinguished triangle in \mathcal{D} . We have to prove that $H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z)$ is exact.

(1) Assume X, Y, Z belong to $\mathcal{D}^{\geq 0}$, and let us show that $0 \rightarrow H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z)$ is exact. Let W be any object of \mathcal{D}^0 . Since $H^0(X) \simeq \tau_{\leq 0} \tau_{\geq 0} X$, by the adjunction isomorphism and Proposition 2.11, then

$$\mathrm{Hom}_{\mathcal{D}^0}(W, H^0(X)) \simeq \mathrm{Hom}_{\mathcal{D}^{\geq 0}}(W, \tau_{\geq 0} X) \simeq \mathrm{Hom}_{\mathcal{D}}(W, X).$$

Similarly, we have $\mathrm{Hom}_{\mathcal{D}^0}(W, H^0(Y)) \simeq \mathrm{Hom}_{\mathcal{D}}(W, Y)$ and $\mathrm{Hom}_{\mathcal{D}^0}(W, H^0(Z)) \simeq \mathrm{Hom}_{\mathcal{D}}(W, Z)$. By TS2, $\mathrm{Hom}_{\mathcal{D}}(W, T^{-1}Z) = 0$. Hence the long exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(W, T^{-1}Z) \rightarrow \mathrm{Hom}_{\mathcal{D}}(W, X) \rightarrow \mathrm{Hom}_{\mathcal{D}}(W, Y) \rightarrow \mathrm{Hom}_{\mathcal{D}}(W, Z),$$

and thus

$$0 \rightarrow \mathrm{Hom}_{\mathcal{D}^0}(W, H^0(X)) \rightarrow \mathrm{Hom}_{\mathcal{D}^0}(W, H^0(Y)) \rightarrow \mathrm{Hom}_{\mathcal{D}^0}(W, H^0(Z))$$

gives the required result.

(2) We only assume $Z \in \mathcal{D}^{\geq 0}$, and we prove that $0 \rightarrow H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z)$ is exact. For any $W \in \mathcal{D}^{< 0}$, we have $\mathrm{Hom}_{\mathcal{D}}(W, Z) = \mathrm{Hom}_{\mathcal{D}}(W, T^{-1}Z) = 0$, and thus $\mathrm{Hom}_{\mathcal{D}}(W, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{D}}(W, Y)$ is an isomorphism. Hence we see that $\tau_{< 0} X \xrightarrow{\tau_{< 0}(f)} \tau_{< 0} Y$ is an isomorphism by the adjunction isomorphism. We apply the octahedral axiom TR5 to $\tau_{< 0} X \xrightarrow{\eta^{-1}(X)} X \xrightarrow{f} Y$, and then we have

$$\begin{array}{ccccccc} \tau_{< 0} X & \xrightarrow{\eta^{-1}(X)} & X & \xrightarrow{\varepsilon^0(X)} & \tau_{\geq 0} X & \xrightarrow{d^0(X)} & T\tau_{< 0} X \\ \parallel & & \downarrow f & & \downarrow u & & \parallel \\ \tau_{< 0} X & \xrightarrow{f \circ \eta^{-1}(X)} & Y & \longrightarrow & V & \longrightarrow & T\tau_{< 0} X \\ \eta^{-1}(X) \downarrow & & \parallel & & \downarrow v & & \downarrow T(\eta^{-1}(X)) \\ X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & TX \\ \varepsilon^0(Y) \downarrow & & \downarrow & & \parallel & & \downarrow \\ \tau_{\geq 0} X & \xrightarrow{u} & V & \xrightarrow{v} & Z & \xrightarrow{w} & T\tau_{\geq 0} X. \end{array}$$

Note that there exists an isomorphism of distinguished triangles:

$$\begin{array}{ccccccc} \tau_{< 0} X & \xrightarrow{f \circ \eta^{-1}(X)} & Y & \longrightarrow & V & \longrightarrow & T\tau_{< 0} X \\ \simeq \downarrow \tau_{< 0}(f) & & \parallel & & \downarrow \simeq & & \simeq \downarrow T(\tau_{< 0}(f)) \\ \tau_{< 0} Y & \xrightarrow{\eta^{-1}(Y)} & Y & \xrightarrow{\varepsilon^0(Y)} & \tau_{\geq 0} Y & \longrightarrow & T\tau_{< 0} Y, \end{array}$$

which is induced naturally from

$$\begin{array}{ccc} \tau_{< 0} X & \xrightarrow{\eta^{-1}(X)} & X \\ \simeq \downarrow \tau_{< 0}(f) & & \downarrow f \\ \tau_{< 0} Y & \xrightarrow{\eta^{-1}(Y)} & Y. \end{array}$$

Hence we have a distinguished triangle

$$\tau_{\geq 0} X \xrightarrow{u} \tau_{\geq 0} Y \xrightarrow{v} Z \xrightarrow{w} T\tau_{\geq 0} X$$

with $\tau_{\geq 0} X, \tau_{\geq 0} Y \in \mathcal{D}^{\geq 0}$. By Proposition 2.14.(i), $H^0(\tau_{\geq 0} X) = H^0(X)$ and $H^0(\tau_{\geq 0} Y) = H^0(Y)$. Therefore by (1) we obtain the required result.

(1') Assume X, Y, Z belong to $\mathcal{D}^{\leq 0}$, and let us show that $H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z) \rightarrow 0$ is exact. Let W be any object of \mathcal{D}^0 . Since $H^0(X) \simeq \tau_{\geq 0} \tau_{\leq 0} X$, by the adjunction

isomorphism and Proposition 2.11, then

$$\mathrm{Hom}_{\mathcal{D}^0}(H^0(X), W) \simeq \mathrm{Hom}_{\mathcal{D}^{\leq 0}}(\tau_{\leq 0} X, W) \simeq \mathrm{Hom}_{\mathcal{D}}(X, W).$$

Similarly, we have $\mathrm{Hom}_{\mathcal{D}^0}(H^0(Y), W) \simeq \mathrm{Hom}_{\mathcal{D}}(Y, W)$ and $\mathrm{Hom}_{\mathcal{D}^0}(H^0(Z), W) \simeq \mathrm{Hom}_{\mathcal{D}}(Z, W)$. By TS2', $\mathrm{Hom}_{\mathcal{D}}(TX, W) = 0$. Hence the long exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(TX, W) \rightarrow \mathrm{Hom}_{\mathcal{D}}(Z, W) \rightarrow \mathrm{Hom}_{\mathcal{D}}(Y, W) \rightarrow \mathrm{Hom}_{\mathcal{D}}(X, W)$$

and thus

$$0 \rightarrow \mathrm{Hom}_{\mathcal{D}^0}(H^0(Z), W) \rightarrow \mathrm{Hom}_{\mathcal{D}^0}(H^0(Y), W) \rightarrow \mathrm{Hom}_{\mathcal{D}^0}(H^0(X), W)$$

gives the required result.

(2') We only assume $X \in \mathcal{D}^{\leq 0}$, and we prove that $H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z) \rightarrow 0$ is exact. For any $W \in \mathcal{D}^{>0}$, we have $\mathrm{Hom}_{\mathcal{D}}(X, W) = \mathrm{Hom}_{\mathcal{D}}(TX, W) = 0$, and thus $\mathrm{Hom}_{\mathcal{D}}(Z, W) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{D}}(Y, W)$ is an isomorphism. Hence we see that $\tau_{>0} Y \xrightarrow{\tau_{>0} g} \tau_{>0} Z$ is an isomorphism by the adjunction isomorphism. We apply the octahedral axiom TR5 to

$$T^{-1} Y \xrightarrow{-T^{-1}(g)} T^{-1} Z \xrightarrow{T^{-1}\varepsilon^1(Z)} T^{-1}\tau_{>0} Z,$$

and then we have

$$\begin{array}{ccccccc} T^{-1} Y & \xrightarrow{-T^{-1}(g)} & T^{-1} Z & \longrightarrow & X & \xrightarrow{f} & Y \\ \parallel & & \downarrow T^{-1}\varepsilon^1(Z) & & \downarrow u & & \parallel \\ T^{-1} Y & \xrightarrow{T^{-1}\varepsilon^1(Z) \circ (-T^{-1}(g))} & T^{-1}\tau_{>0} Z & \longrightarrow & U & \longrightarrow & Y \\ \downarrow -T^{-1}(g) & & \parallel & & \downarrow v & & \downarrow -g \\ T^{-1} Z & \xrightarrow{T^{-1}\varepsilon^1(Z)} & T^{-1}\tau_{>0} Z & \longrightarrow & \tau_{\leq 0} Z & \longrightarrow & Z \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ X & \xrightarrow{\quad u \quad} & U & \xrightarrow{\quad v \quad} & \tau_{\leq 0} Z & \xrightarrow{\quad w \quad} & TX. \end{array}$$

Note that there exists an isomorphism of distinguished triangles:

$$\begin{array}{ccccccc} T^{-1} Y & \xrightarrow{T^{-1}\varepsilon^1(Y)} & T^{-1}\tau_{>0} Y & \longrightarrow & \tau_{\leq 0} Y & \longrightarrow & Y \\ \parallel & & \simeq \downarrow -T^{-1}\tau_{>0}(g) & & \downarrow \simeq & & \parallel \\ T^{-1} Y & \xrightarrow{T^{-1}\varepsilon^1(Z) \circ (-T^{-1}(g))} & T^{-1}\tau_{>0} Z & \longrightarrow & U & \longrightarrow & Y, \end{array}$$

which is induced naturally from

$$\begin{array}{ccc} Y & \xrightarrow{\varepsilon^1(Y)} & \tau_{>0} Y \\ \downarrow g & & \downarrow \tau_{>0}(g) \\ Z & \xrightarrow{\varepsilon^1(Z)} & \tau_{>0} Z. \end{array}$$

Hence we have a distinguished triangle

$$X \xrightarrow{u} \tau_{\leq 0} Y \xrightarrow{v} \tau_{\leq 0} Z \xrightarrow{w} TX$$

with $\tau_{\leq 0} Y, \tau_{\leq 0} Z \in \mathcal{D}^{\leq 0}$. By Proposition 2.14.(i), $H^0(\tau_{\leq 0} Y) = H^0(Y)$ and $H^0(\tau_{\leq 0} Z) = H^0(Z)$. Therefore by (1') we obtain the required result.

(3) We shall prove the general case. We apply the octahedral axiom TR5 to

$$\tau_{\leq 0} X \xrightarrow{\eta^0(X)} X \xrightarrow{f} Y.$$

Then we have

$$\begin{array}{ccccccc}
 \tau_{\leq 0} X & \xrightarrow{\eta^0(X)} & X & \xrightarrow{\varepsilon^1(X)} & \tau_{\geq 1} X & \xrightarrow{d^1(X)} & T\tau_{\leq 0} X \\
 \parallel & & \downarrow f & & \downarrow u & & \parallel \\
 \tau_{\leq 0} X & \xrightarrow{f \circ \eta^0(X)} & Y & \longrightarrow & Q & \longrightarrow & T\tau_{\leq 0} X \\
 \eta^0(X) \downarrow & & \parallel & & \downarrow v & & \downarrow T(\eta^0 X) \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX \\
 \varepsilon^1(Y) \downarrow & & \downarrow & & \parallel & & \downarrow \\
 \tau_{\geq 1} X & \xrightarrow{u} & Q & \xrightarrow{v} & Z & \xrightarrow{w} & T\tau_{\geq 1} X.
 \end{array}$$

Hence we obtain the following two distinguished triangles:

$$\tau_{\leq 0} X \longrightarrow Y \longrightarrow Q \longrightarrow T\tau_{\leq 0} X$$

and

$$Q \longrightarrow Z \longrightarrow T\tau_{\geq 1} X \longrightarrow TQ.$$

Since $\tau_{\leq 0} X \in \mathcal{D}^{\leq 0}$, it follows from (2') and $H^0(\tau_{\leq 0} X) \simeq H^0(X)$ that there exists an exact sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \longrightarrow H^0(Q) \longrightarrow 0.$$

Since $T\tau_{\geq 1} X \in \mathcal{D}^{\geq 0}$, it follows from (2) that there exists an exact sequence

$$0 \rightarrow H^0(Q) \rightarrow H^0(Z).$$

Therefore we obtain $H^0(Q) \simeq \text{Coker}(H^0(f))$ and the required exact sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z).$$

This completes the proof. □

4 Bounded t-structures

Lemma 4.1. *We have the following properties:*

$$(4.1) \quad H^i(\tau_{\geq a} X) = \begin{cases} 0 & (i < a) \\ H^i(X) & (i \geq a) \end{cases}$$

$$(4.2) \quad H^i(\tau_{\leq b} X) = \begin{cases} H^i(X) & (i \leq b) \\ 0 & (i > b) \end{cases}$$

Proof. This follows from Proposition 2.14.

(4.1). If $i < a$, then $H^i(\tau_{\geq a} X) = T^i \tau_{\geq i} \tau_{\leq i} \tau_{\geq a} X = 0$ and if $i \geq a$, then

$$H^i(\tau_{\geq a} X) = T^i \tau_{\leq i} \tau_{\geq i} \tau_{\geq a} X = T^i \tau_{\leq i} \tau_{\geq i} X = H^i(X).$$

(4.2). If $i \leq b$, then

$$H^i(\tau_{\leq b} X) = T^i \tau_{\geq i} \tau_{\leq i} \tau_{\leq b} X = T^i \tau_{\geq i} \tau_{\leq i} X = H^i(X)$$

and if $i > b$, then $H^i(\tau_{\leq b} X) = T^i \tau_{\leq i} \tau_{\geq i} \tau_{\leq b} X = 0$. □

Proposition 4.2. *Let X be an object of \mathcal{D} .*

(i) *Assume $X \in \mathcal{D}^{\geq a}$ for some integer a . Then $X \in \mathcal{D}^{\geq 0}$ if and only if $H^n(X) = 0$ for $n < 0$.*

(ii) *Assume $X \in \mathcal{D}^{\leq a}$ for some integer a . Then $X \in \mathcal{D}^{\leq 0}$ if and only if $H^n(X) = 0$ for $n > 0$.*

Proof. (i) If X belongs to $\mathcal{D}^{\geq 0}$, then by Proposition 2.11, we have $X \simeq \tau^{\geq 0} X$. Hence by Lemma 4.1, we obtain $H^n(X) = 0$ for $n < 0$.

Conversely, suppose that $H^n(X) = 0$ for $n < 0$. Then we may assume that $a < 0$. By Proposition 2.14, we have the distinguished triangle:

$$(4.3) \quad T^{-n} H^n(X) \longrightarrow \tau_{\geq n} X \longrightarrow \tau_{\geq n+1} X \longrightarrow T^{-n+1} H^n(X).$$

Hence $X \simeq \tau_{\geq a} X \simeq \tau_{\geq a+1} X \simeq \cdots \simeq \tau_{\geq 0} X$. Therefore $X \in \mathcal{D}^{\geq 0}$.

(ii) If X belongs to $\mathcal{D}^{\leq 0}$, then by Proposition 2.11, we have $X \simeq \tau_{\leq 0} X$. Hence by Lemma 4.1, we obtain $H^n(X) = 0$ for $n > 0$.

Conversely, suppose that $H^n(X) = 0$ for $n > 0$. Then we may assume that $a > 0$. By Proposition 2.14, we have the distinguished triangle:

$$(4.4) \quad \tau_{\leq n-1} X \longrightarrow \tau_{\leq n} X \longrightarrow T^{-n} H^n(X) \longrightarrow T \tau_{\leq n-1} X.$$

Hence $X \simeq \tau_{\leq a} X \simeq \tau_{\leq a-1} X \simeq \cdots \simeq \tau_{\leq 0} X$. Therefore $X \in \mathcal{D}^{\leq 0}$. □

Definition 4.3. A t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category (\mathcal{D}, T) is said to be **nondegenerate** if

$$\bigcap_{n=-\infty}^{\infty} \mathcal{D}^{\leq n} = \{0\} \quad \text{and} \quad \bigcap_{n=-\infty}^{\infty} \mathcal{D}^{\geq n} = \{0\},$$

where 0 is the zero object in \mathcal{D} . □

Proposition 4.4. *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a nondegenerate t -structure on a triangulated category (\mathcal{D}, T) . Then we have the following:*

(i) *An object $X \in \mathcal{D}$ such that $H^i(X) = 0$ for all $i \in \mathbb{Z}$ is the zero object.*

(ii) *A morphism $f: X \rightarrow Y$ in \mathcal{D} is an isomorphism if and only if $H^i(f)$ are isomorphisms for all $i \in \mathbb{Z}$ in the heart \mathcal{D}^0 of the t -structure.*

(iii) (a) $\mathcal{D}^{\leq n} = \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for all } i > n\}$.

(b) $\mathcal{D}^{\geq n} = \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for all } i < n\}$.

Proof. (i) If $X \in \mathcal{D}^{\leq 0}$, by the distinguished triangle (4.4), we get

$$X \simeq \tau_{\leq 0} X \simeq \tau_{\leq -1} X \simeq \tau_{\leq -2} X \simeq \cdots \simeq \tau_{\leq -n} X \simeq \cdots.$$

Hence $X \in \bigcap_{n=-\infty}^{\infty} \mathcal{D}^{\leq n} = \{0\}$.

If $X \in \mathcal{D}^{\geq 1}$, by the distinguished triangle (4.3), we get

$$X \simeq \tau_{\geq 1} X \simeq \tau_{\geq 2} X \simeq \tau_{\geq 3} X \simeq \cdots \simeq \tau_{\geq n} X \simeq \cdots.$$

Hence $X \in \bigcap_{n=-\infty}^{\infty} \mathcal{D}^{\geq n} = \{0\}$.

The general case follows from Lemma 4.1. Indeed, from the lemma, we have $H^i(\tau_{\leq 0} X) = H^i(\tau_{\geq 1} X) = 0$ for all i . Then by the above results, we obtain $\tau_{\leq 0} X = \tau_{\geq 1} X = 0$. Therefore the distinguished triangle (2.1) yields $X = 0$.

(ii) Assume $H^i(f)$ are isomorphisms for all i . We embed the morphism $f: X \rightarrow Y$ into a distinguished triangle $X \xrightarrow{f} Y \rightarrow Z \rightarrow TX$. Then we have $H^i(Z) = 0$ for all i since H^0 is a cohomological functor and $H^i(f)$ are isomorphisms. Hence by (i), we obtain $Z = 0$ and f is an isomorphism. The converse is clear.

(iii) First we show the case $n = 0$.

(a) If $H^i(X) = 0$ for all $i > 0$, then $H^i(\tau_{\geq 1} X) = 0$ for all i by Lemma 4.1, and so $\tau_{\geq 1} X = 0$ by (i). Hence the distinguished triangle (2.1) yields $X \simeq \tau_{\leq 0} X \in \mathcal{D}^{\leq 0}$. Conversely, if $X \in \mathcal{D}^{\leq 0}$, then $X \simeq \tau_{\leq 0} X$, and thus $\tau_{\geq 1} X = 0$. Hence by Lemma 4.1 we obtain $H^i(X) = H^i(\tau_{\geq 1} X) = 0$ for all $i > 0$.

(b) If $H^i(X) = 0$ for all $i < 0$, then $H^i(\tau_{\leq 0} T^{-1} X) = 0$ for all i by Lemma 4.1, and so $T^{-1} \tau_{\leq -1} X \simeq \tau_{\leq 0} T^{-1} X = 0$ by (i). Hence $\tau_{\leq -1} X = 0$, and the distinguished triangle (2.1) yields $X \simeq \tau_{\geq 0} X \in \mathcal{D}^{\geq 0}$. Conversely, if $X \in \mathcal{D}^{\geq 0}$, then $X \simeq \tau_{\geq 0} X$, and thus $\tau_{\leq -1} X = 0$. Hence by Lemma 4.1 we obtain

$$H^i(X) = H^{i+1}(T^{-1} X) = H^{i+1}(\tau_{\leq 0} T^{-1} X) = H^{i+1}(T^{-1} \tau_{\leq -1} X) = 0$$

for all $i < 0$.

For the general case, we take TX for X . □

Definition 4.5. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on a triangulated category (\mathcal{D}, T) and \mathcal{D}^0 the heart of the t -structure. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is said to be **bounded** if it is nondegenerate and for any $X \in \mathcal{D}$, only a finite number of objects $H^i(X) \in \mathcal{D}^0$ is nonzero. □

Lemma 4.6. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on a triangulated category (\mathcal{D}, T) . Then the following two conditions are equivalent.

(i) $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is bounded.

(ii) For each $X \in \mathcal{D}$, there exist integers n, m such that $X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$, i.e.,

$$\mathcal{D} = \bigcup_{n, m \in \mathbb{Z}} \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}.$$

Proof. (i) \Rightarrow (ii). Let $X \in \mathcal{D}$ and set

$$m := \min\{i \in \mathbb{Z} \mid H^i(X) \neq 0\} \quad \text{and} \quad n := \max\{i \in \mathbb{Z} \mid H^i(X) \neq 0\}.$$

Then $X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$.

(ii) \Rightarrow (i). Let $X \in \bigcap_{n=-\infty}^{\infty} \mathcal{D}^{\leq n}$. Then it follows from (ii) that there exist an integers m_0 such that $X \in \mathcal{D}^{\geq m_0}$. Since $X \in \mathcal{D}^{\leq n}$ for all n , $X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m_0}$ for all n . Hence $X \simeq \tau^{\geq m_0} X \simeq \tau^{\leq n} \tau^{\geq m_0} X = 0$ for all $n < m_0$, and thus $\bigcap_{n=-\infty}^{\infty} \mathcal{D}^{\leq n} = \{0\}$. Similarly, we obtain $\bigcap_{n=-\infty}^{\infty} \mathcal{D}^{\geq n} = \{0\}$. Therefore $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is nondegenerate. Moreover, since there exist integers n_0, m_0 such that $X \in \mathcal{D}^{\leq n_0} \cap \mathcal{D}^{\geq m_0}$, we see that $H(X)^i = 0$ if $i \notin [m_0, n_0]$. Therefore $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is bounded. \square

Lemma 4.7. *Let (\mathcal{D}, T) be a triangulated category. Assume that for every nonzero object $X \in \mathcal{D}$, there exists a collection of finite distinguished triangles*

$$0 = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \rightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n = X.$$

For $0 \leq i \leq j \leq n - 1$, consider distinguished triangles

$$X_i \xrightarrow{f_j \circ \cdots \circ f_i} X_{j+1} \longrightarrow X_{j+1}^i \longrightarrow TX_i.$$

Then X_{j+1}^i has the following decomposition into a collection of finite distinguished triangles

$$0 = X_i^i \longrightarrow X_{i+1}^i \longrightarrow X_{i+2}^i \rightarrow \cdots \rightarrow X_j^i \longrightarrow X_{j+1}^i.$$

Proof. Put $g_{i,j} := f_j \circ \cdots \circ f_i$ for $0 \leq i \leq j \leq n - 1$. We use induction on $k := j - i$. If $k = 0$, then $g_{i,i} = f_i$ and we have the following isomorphism.

$$\begin{array}{ccccccc} X_i & \xrightarrow{g_{i,i}} & X_{i+1} & \longrightarrow & X_{i+1}^i & \longrightarrow & TX_i \\ \parallel & & \parallel & & \downarrow \simeq & & \parallel \\ X_i & \xrightarrow{f_i} & X_{i+1} & \longrightarrow & A_{i+1} & \longrightarrow & TX_i \end{array}$$

Hence we obtain a distinguished triangle

$$0 = X_i^i \longrightarrow X_{i+1}^i \longrightarrow A_{i+1} \longrightarrow X_i^i.$$

Assume that the statement holds until $k - 1$. By induction hypothesis, X_{i+k-1}^i has the following decomposition

$$0 = X_i^i \longrightarrow X_{i+1}^i \longrightarrow X_{i+2}^i \longrightarrow \cdots \longrightarrow X_{i+k-2}^i \longrightarrow X_{i+k-1}^i .$$

Now we have the following octahedron diagram.

$$\begin{array}{ccccccc}
 X_i & \xrightarrow{g_{i,i+k-2}} & X_{i+k-1} & \longrightarrow & X_{i+k-1}^i & \longrightarrow & TX_i \\
 \parallel & & \downarrow f_{i+k-1} & & \downarrow & & \parallel \\
 X_i & \xrightarrow{g_{i,i+k-1}} & X_{i+k} & \longrightarrow & X_{i+k}^i & \longrightarrow & TX_i \\
 \downarrow g_{i,i+k-2} & & \parallel & & \downarrow & & \downarrow \\
 X_{i+k-1} & \xrightarrow{f_{i+k-1}} & X_{i+k} & \longrightarrow & A_{i+k} & \longrightarrow & TX_{i+k-1} \\
 \downarrow & & \downarrow & & \parallel & & \downarrow \\
 X_{i+k-1}^i & \cdots \longrightarrow & X_{i+k}^i & \cdots \longrightarrow & A_{i+k} & \cdots \longrightarrow & TX_{i+k-1}^i
 \end{array}$$

This diagram yields the distinguished triangle

$$\begin{array}{ccc}
 X_{i+k-1}^i & \longrightarrow & X_{i+k}^i \\
 & \swarrow \text{dashed} & \searrow \text{solid} \\
 & & A_{i+k}
 \end{array}$$

Therefore we obtain the required decomposition. \square

Lemma 4.8. *Let (\mathcal{D}, T) be a triangulated category and \mathcal{A} a strictly full additive subcategory of \mathcal{D} . Assume that the following two conditions hold:*

- (i) *if $k_1 > k_2$ are integers, then $\text{Hom}_{\mathcal{D}}(T^{k_1} A, T^{k_2} B) = 0$ for all $A, B \in \mathcal{A}$.*
- (ii) *for every nonzero object $X \in \mathcal{D}$, there exists a finite sequence of integers*

$$k_1 > k_2 > \cdots > k_n$$

and a collection of distinguished triangles

$$0 = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n = X$$

with nonzero objects $A_i \in T^{k_i} \mathcal{A}$ for all i .

Then we have the following.

- (1) for any integer $k > k_1$ and any object $E \in T^k \mathcal{A}$, we have $\text{Hom}(E, X_i) = 0$ for $0 \leq i \leq n$. In particular, $\text{Hom}(E, X) = 0$.
- (2) for any object $F \in T^{k_1} \mathcal{A}$, the map $(f_{n-1} \circ \cdots \circ f_1)_* : \text{Hom}(F, X_1) \rightarrow \text{Hom}(F, X)$ is injective. In particular, the morphism $f_{n-1} \circ \cdots \circ f_1 : X_1 \rightarrow X$ is nonzero, and then each f_i ($i = 1, 2, \dots, n - 1$) is nonzero.

Proof. (1) We use induction on i . If $i = 0$, then it is clear since $X_0 = 0$. Assume that the statement holds for less than $i > 0$. Since $k > k_1 \geq k_i$ and $A_i \in T^{k_i} \mathcal{A}$, by the condition (i), we have $\text{Hom}(E, A_i) = 0$. Hence the exact sequence

$$\text{Hom}(E, X_{i-1}) \xrightarrow{(f_{i-1})_*} \text{Hom}(E, X_i) \longrightarrow \text{Hom}(E, A_i)$$

yields $\text{Hom}(E, X_i) = 0$.

(2) We take the objects X_i^1 ($i = 1, 2, \dots, n$) as in Lemma 4.7. It is enough to show that $\text{Hom}(F, T^{-1} X_n^1) = 0$. By Lemma 4.7, the object X_n^1 has the following decomposition into distinguished triangles

$$0 = X_1^1 \longrightarrow X_2^1 \longrightarrow X_3^1 \longrightarrow \cdots \longrightarrow X_{n-1}^1 \longrightarrow X_n^1,$$

and then we have the decomposition of $T^{-1} X_n^1$

$$0 = T^{-1} X_1^1 \longrightarrow T^{-1} X_2^1 \longrightarrow \cdots \longrightarrow T^{-1} X_{n-1}^1 \longrightarrow T^{-1} X_n^1,$$

with $T^{-1} A_i \in T^{k_i-1} \mathcal{A}$ for $2 \leq i \leq n$. Hence by (1) we have $\text{Hom}(F, T^{-1} X_n^1) = 0$ since $k_1 > k_i > k_i - 1$.

In particular, since $X_1 \simeq A_1 \neq 0$, $\text{Hom}(X_1, X_1) \neq 0$ and the morphism $f_{n-1} \circ \cdots \circ f_1 = (f_{n-1} \circ \cdots \circ f_1)_*(\text{id}_{X_1}) : X_1 \rightarrow X$ is nonzero. \square

Lemma 4.9. Let (\mathcal{D}, T) be a triangulated category and let \mathcal{A} be a strictly full additive subcategory of \mathcal{D} . Assume that the following two conditions hold:

- (i) if $k_1 > k_2$ are integers, then $\text{Hom}_{\mathcal{D}}(T^{k_1} A, T^{k_2} B) = 0$ for all $A, B \in \mathcal{A}$.
- (ii) for every nonzero object $X \in \mathcal{D}$, there exists a finite sequence of integers

$$k_1 > k_2 > \cdots > k_n$$

and a collection of distinguished triangles

$$0 = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n = X$$

with nonzero objects $A_i \in T^{k_i} \mathcal{A}$ for $1 \leq i \leq n$.

Then the integer n and the sequence (k_1, k_2, \dots, k_n) is uniquely determined by X . Moreover, the sequence (A_1, A_2, \dots, A_n) is also uniquely determined by X up to isomorphisms.

Proof. Suppose that for a nonzero object $X \in \mathcal{D}$, there exists an another finite sequence of integers

$$k'_1 > k'_2 > \dots > k'_m$$

and a collection of distinguished triangles

$$0 = X'_0 \xrightarrow{f'_0} X'_1 \xrightarrow{f'_1} X'_2 \rightarrow \dots \rightarrow X'_{m-1} \xrightarrow{f'_{m-1}} X'_m = X$$

with nonzero objects $A'_j \in T^{k'_j} \mathcal{A}$ for $1 \leq j \leq m$.

Assume that $k_1 > k'_1$. By Lemma 4.8.(1), we have $\text{Hom}(X_1, X) = 0$. On the other hand, by Lemma 4.8.(2), we have $\text{Hom}(X_1, X) \neq 0$. This is a contradiction. Hence $k_1 \leq k'_1$. Similarly, if we assume $k'_1 > k_1$, then we have a contradiction and $k'_1 \leq k_1$. Therefore we obtain $k_1 = k'_1$.

Next we define objects X_n^1 and $X'_m{}^1$ such that

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_{n-1} \circ \dots \circ f_1} & X & \longrightarrow & X_n^1 & \longrightarrow & TX_1 \\ & & \parallel & & & & \\ X'_1 & \xrightarrow{f'_{m-1} \circ \dots \circ f'_1} & X & \longrightarrow & X'_m{}^1 & \longrightarrow & TX'_1 \end{array}$$

are distinguished triangles. By Lemma 4.7, we have the following two decompositions

$$0 = X_1^1 \longrightarrow X_2^1 \rightarrow \dots \rightarrow X_{n-1}^1 \longrightarrow X_n^1$$

and

$$0 = X'^1_1 \longrightarrow X'^1_2 \rightarrow \dots \rightarrow X'^1_{m-1} \longrightarrow X'^1_m.$$

By Lemma 4.8.(1), we obtain $\text{Hom}(X_1, X'^1_m) = 0$ since $k_1 = k'_1 > k'_2$. Then there exists a morphism $\varphi : X_1 \rightarrow X'_1$ such that

$$\begin{array}{ccccccc}
 X_1 & \xrightarrow{f_{n-1} \circ \cdots \circ f_1} & X & \longrightarrow & X_n^1 & \longrightarrow & TX_1 \\
 \varphi \downarrow & & \parallel & & & & \\
 X'_1 & \xrightarrow{f'_{m-1} \circ \cdots \circ f'_1} & X & \longrightarrow & X'_m{}^1 & \longrightarrow & TX'_1
 \end{array}$$

Similarly, there exists a morphism $\varphi' : X'_1 \rightarrow X_1$ such that

$$\begin{array}{ccccccc}
 X'_1 & \xrightarrow{f'_{m-1} \circ \cdots \circ f'_1} & X & \longrightarrow & X'_m{}^1 & \longrightarrow & TX'_1 \\
 \varphi' \downarrow & & \parallel & & & & \\
 X_1 & \xrightarrow{f_{n-1} \circ \cdots \circ f_1} & X & \longrightarrow & X_n^1 & \longrightarrow & TX_1
 \end{array}$$

Hence we have $(f_{n-1} \circ \cdots \circ f_1)_*(\varphi' \circ \varphi) = (f_{n-1} \circ \cdots \circ f_1)_*(\text{id}_{X_1})$. By Lemma 4.8.(2), we obtain $\varphi' \circ \varphi = \text{id}_{X_1}$. Similarly, we obtain $\varphi \circ \varphi' = \text{id}_{X'_1}$. Therefore $\varphi : X_1 \rightarrow X'_1$ is an isomorphism and $A_1 \simeq X_1 \simeq X'_1 \simeq A'_1$. Moreover, from the following isomorphism of distinguished triangles

$$\begin{array}{ccccccc}
 X_1 & \xrightarrow{f_{n-1} \circ \cdots \circ f_1} & X & \longrightarrow & X_n^1 & \longrightarrow & TX_1 \\
 \varphi \downarrow \simeq & & \parallel & & \downarrow \simeq & & T\varphi \downarrow \simeq \\
 X'_1 & \xrightarrow{f'_{m-1} \circ \cdots \circ f'_1} & X & \longrightarrow & X'_m{}^1 & \longrightarrow & TX'_1
 \end{array}$$

we obtain $X_n^1 \simeq X'_m{}^1$.

For $i \geq 2$, we obtain inductively that $k_i = k'_i$ and the following isomorphism of distinguished triangles

$$\begin{array}{ccccccc}
 X_i^{i-1} & \longrightarrow & X_n^{i-1} & \longrightarrow & X_n^i & \longrightarrow & TX_i^{i-1} \\
 \varphi_i \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & T\varphi_i \downarrow \simeq \\
 X_i^{i-1} & \longrightarrow & X_m^{i-1} & \longrightarrow & X_m^i & \longrightarrow & TX_i^{i-1}
 \end{array}$$

Therefore we have $A_i \simeq X_i^{i-1} \simeq X_n^{i-1} \simeq A'_i$ and $X_n^i \simeq X_m^i$.

Let us show that $n = m$. If $n < m$, then $k_i = k'_i$ and $A_i \simeq A'_i$ for $i = 1, 2, \dots, n$, and we have a collection of distinguished triangles

$$\begin{array}{ccccccc}
 0 = X_n^m & \longrightarrow & X_{n+1}^m & \longrightarrow & \cdots & \longrightarrow & X_{m-1}^m & \longrightarrow & X_m^m & \simeq & X_n^n = 0 \\
 & & \swarrow & & & & \swarrow & & & & \\
 & & A'_{n+1} & & & & A'_m & & & &
 \end{array}$$

with nonzero objects $A'_j \in T^{k'_j} \mathcal{A}$ for all j . On the other hand, by Lemma 4.8.(2), the morphism $0 \neq A'_{n+1} \simeq X'_{n+1} \rightarrow X_n^m$ is nonzero, and then $X_n^m \neq 0$. This is a contradiction. Hence we obtain $m \leq n$. Similarly we obtain $n \leq m$. Therefore $n = m$. \square

If the conditions (i) and (ii) in Lemma 4.9 hold, then for each nonzero object $X \in \mathcal{D}$, the integer n and the sequence (k_1, k_2, \dots, k_n) are determined uniquely. Then we put

$$k_{\max}(X) := k_1 \quad \text{and} \quad k_{\min}(X) := k_n.$$

Note that $k_{\max}(X)$ and $k_{\min}(X)$ have the following properties:

$$k_{\max}(T^k X) = k_{\max}(X) + k \quad \text{and} \quad k_{\min}(T^k X) = k_{\min}(X) + k$$

for any integers k .

Proposition 4.10. *Let (\mathcal{D}, T) be a triangulated category and let \mathcal{A} be a strictly full additive subcategory of \mathcal{D} . Then \mathcal{A} is the heart of a bounded t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on (\mathcal{D}, T) if and only if the following two conditions hold:*

(i) *if $k_1 > k_2$ are integers, then $\text{Hom}_{\mathcal{D}}(T^{k_1} A, T^{k_2} B) = 0$ for all $A, B \in \mathcal{A}$.*

(ii) *for every nonzero object $X \in \mathcal{D}$, there exists a finite sequence of integers*

$$k_1 > k_2 > \dots > k_n$$

and a collection of distinguished triangles

$$\begin{array}{ccccccc}
 0 = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow \cdots & \longrightarrow X_{n-1} & \longrightarrow & X_n = X \\
 & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
 & & A_1 & & A_2 & & A_n & &
 \end{array}$$

with nonzero objects $A_i \in T^{k_i} \mathcal{A}$ for all i .

Proof. “only if” part: Assume that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a bounded t -structure on (\mathcal{D}, T) and \mathcal{A} is the heart of the t -structure.

(i) We have $T^{k_1} A \in T^{k_1} \mathcal{A} \subset \mathcal{D}^{\leq -k_1}$, $T^{k_2} B \in T^{k_2} \mathcal{A} \subset \mathcal{D}^{\geq -k_2}$ and $-k_1 < -k_2$. Hence we obtain $\text{Hom}_{\mathcal{D}}(T^{k_1} A, T^{k_2} B) = 0$.

(ii) Let $X \in \mathcal{D}$ be a nonzero object. It follows from Lemma 4.6 that there exist integers $a \leq b$ such that $X \in \mathcal{D}^{\leq b} \cap \mathcal{D}^{\geq a}$. If $a = b$, then we have $X \in T^{-a} \mathcal{A}$ and a distinguished triangle $0 \longrightarrow X \xrightarrow{\text{id}_X} X \longrightarrow 0$. If $a < b$, then we obtain a collection of distinguished triangles

$$\begin{array}{ccccccc}
 0 \simeq \tau_{\leq a-1} X & \longrightarrow & \tau_{\leq a} X & \longrightarrow & \tau_{\leq a+1} X & \longrightarrow \cdots & \longrightarrow \tau_{\leq b-1} X & \longrightarrow & X \\
 & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
 & & T^{-a} H^b(X) & & T^{-a-1} H^{a+1}(X) & & T^{-b} H^b(X) & &
 \end{array}$$

with $-a > -a-1 > \dots > -b$.

“if” part: Assume that the conditions (i) and (ii) hold. First of all, we define strictly full subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ of \mathcal{D} as

$$\mathcal{D}^{\leq n} := \{X \in \mathcal{D} \mid X \neq 0, k_{\min}(X) \geq -n\} \cup \{0\},$$

$$\mathcal{D}^{\geq n} := \{X \in \mathcal{D} \mid X \neq 0, k_{\max}(X) \leq -n\} \cup \{0\}.$$

Then we see that $\mathcal{D}^{\leq n} = T^{-n} \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq n} = T^{-n} \mathcal{D}^{\geq 0}$. Indeed, for any nonzero object $X \in \mathcal{D}$, we have

in the condition (ii), we have $X_i \in \mathcal{D}^{\leq 0}$.

Now we consider the distinguished triangle

$$X_i \xrightarrow{f_{n-1} \circ \dots \circ f_i} X \longrightarrow X_n^i \longrightarrow TX_i$$

as in Lemma 4.7, and then we have $X_n^i \in \mathcal{D}^{\geq 1}$. Therefore $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t -structure on \mathcal{D} .

Next the t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is bounded, because for any nonzero object $X \in \mathcal{D}$, we have $X \in \mathcal{D}^{\leq -k_{\min}(X)} \cap \mathcal{D}^{\geq -k_{\max}(X)}$.

Finally \mathcal{A} is the heart of the t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Indeed, it follows from the definition of $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ that $\mathcal{A} \subseteq \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. If X is a nonzero object in $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, then $X \simeq A_1 \in \mathcal{A}$ since $k_{\min}(X) = k_{\max}(X) = 0$. Hence we obtain $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. \square

Remark 4.11. Let (\mathcal{D}, T) be a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a bounded t -structure on \mathcal{D} with the heart \mathcal{D}^0 . Then

- $\mathcal{D}^{\leq n}$ is the smallest strictly full extension-closed subcategory of \mathcal{D} that contains all the objects of the subcategories $T^k \mathcal{D}^0$ for integers $k \geq -n$,
- $\mathcal{D}^{\geq n}$ is the smallest strictly full extension-closed subcategory of \mathcal{D} that contains all the objects of the subcategories $T^k \mathcal{D}^0$ for integers $k \leq -n$.

Indeed, let $\mathcal{D}'^{\leq n}$ (resp. $\mathcal{D}'^{\geq n}$) be the smallest strictly full extension-closed subcategory of \mathcal{D} that contains all the objects of the subcategories $T^k \mathcal{D}^0$ for integers $k \geq -n$ (resp. $k \leq -n$). Then it follows from Lemma 4.10 that $\mathcal{D}^{\leq n} \subseteq \mathcal{D}'^{\leq n}$ and $\mathcal{D}^{\geq n} \subseteq \mathcal{D}'^{\geq n}$. Since $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$ is a t -structure, $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ are strictly full extension-closed subcategories of \mathcal{D} . Hence $\mathcal{D}^{\leq n} = \mathcal{D}'^{\leq n}$ and $\mathcal{D}^{\geq n} = \mathcal{D}'^{\geq n}$. \square

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