

<研究ノート>

Triangulated Categories. II: Localization of Categories

寺川 宏之 大野 真裕

TERAKAWA Hiroyuki, OHNO Masahiro

要旨

この研究ノートは、主に代数幾何学に現れる三角圏や導来圏の性質についてまとめた一連の研究ノートの第2部である。今回は、代数幾何学においてホモトピー圏から導来圏を構成するために用いられる重要な概念である圏の局所化に関する概説である。

第2節において圏の局所化の定義を与える。第3節では圏の局所化を構成するために必要となる乗法系の定義とその基本性質を解説する。第4節では与えられた圏の乗法系から局所化と呼ばれる新しい圏を構成する。最後に、第5節において飽和乗法系について解説する。

1 Introduction

This exposition is the second part of our study of triangulated categories in algebraic geometry. We shall explain the definition and fundamental properties of a localization of a category and describe their proofs in detail. Main reference is Kashiwara and Schapira [KS06].

2 Definition of Localization

Let \mathcal{C} be a category and let \mathcal{S} be a family of morphisms in \mathcal{C} .

Definition 2.1. A **localization** of \mathcal{C} by \mathcal{S} is the data of a category $\mathcal{C}_{\mathcal{S}}$ and a functor $Q_{\mathcal{S}}: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{S}}$, called a **localization functor**, satisfying the conditions:

- L1 for all $s \in \mathcal{S}$, $Q_{\mathcal{S}}(s)$ is an isomorphism,
- L2 for any category \mathcal{D} and any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exist a functor $F_{\mathcal{S}}: \mathcal{C}_{\mathcal{S}} \rightarrow \mathcal{D}$ and an isomorphism $F \simeq F_{\mathcal{S}} \circ Q_{\mathcal{S}}$,
- L3 if G_1 and G_2 are two objects of $\text{Fct}(\mathcal{C}_{\mathcal{S}}, \mathcal{D})$, then the natural map

$\text{Hom}_{\text{Fct}(\mathcal{C}_S, \mathcal{D})}(G_1, G_2) \xrightarrow{Q_S^*} \text{Hom}_{\text{Fct}(\mathcal{C}, \mathcal{D})}(G_1 \circ Q_S, G_2 \circ Q_S)$
is bijective. □

Proposition 2.2. *We have the following.*

- (i) F_S in L2 is unique up to unique isomorphism.
- (ii) If \mathcal{C}_S exists, it is unique up to equivalence of categories.

Proof. (i) Let $G : \mathcal{C}_S \rightarrow \mathcal{D}$ be another functor admitting an isomorphism $F \simeq G \circ Q_S$. Then we have isomorphisms $f : F_S \circ Q_S \xrightarrow{\sim} G \circ Q_S$ and $g : G \circ Q_S \xrightarrow{\sim} F_S \circ Q_S$ such that $f \circ g = \text{id}_{G \circ Q_S}$ and $g \circ f = \text{id}_{F_S \circ Q_S}$. Since $Q^* := Q_S^*$ is surjective, there exist morphisms $\theta : F_S \rightarrow G$ and $\lambda : G \rightarrow F_S$ such that $Q^*(\theta) = f$ and $Q^*(\lambda) = g$. Then we have

$$Q^*(\theta \circ \lambda) = Q^*(\theta) \circ Q^*(\lambda) = f \circ g = \text{id}_{G \circ Q_S} = Q^*(\text{id}_G)$$

and

$$Q^*(\lambda \circ \theta) = Q^*(\lambda) \circ Q^*(\theta) = g \circ f = \text{id}_{F_S \circ Q_S} = Q^*(\text{id}_{F_S}).$$

Since Q^* is injective, $\theta \circ \lambda = \text{id}_G$ and $\lambda \circ \theta = \text{id}_{F_S}$. Hence we obtain an isomorphism $\theta : F_S \simeq G$.

(ii) Let (\mathcal{C}'_S, Q'_S) be another localization of \mathcal{C} by \mathcal{S} . By L1, $Q'_S(s)$ is an isomorphism for $s \in \mathcal{S}$.

Then by L2, we have $F \circ Q_S \simeq Q'_S$ and $G \circ Q'_S \simeq Q_S$, and thus

$$G \circ F \circ Q_S \simeq Q_S = \text{id}_{\mathcal{C}_S} \circ Q_S \text{ and } F \circ G \circ Q'_S \simeq Q'_S = \text{id}_{\mathcal{C}'_S} \circ Q'_S.$$

By (i), we obtain $G \circ F \simeq \text{id}_{\mathcal{C}_S}$ and $F \circ G \simeq \text{id}_{\mathcal{C}'_S}$. Hence \mathcal{C}'_S is equivalent to \mathcal{C}_S . □

3 Multiplicative Systems

Definition 3.1. Let \mathcal{S} be a family of morphisms in \mathcal{C} . Then

- (i) \mathcal{S} is called a **left multiplicative system** in \mathcal{C} if it satisfies the axioms MS1, MS2, LMS3 and LMS4 below, and
- (ii) \mathcal{S} is called a **right multiplicative system** in \mathcal{C} if it satisfies the axioms MS1, MS2, RMS3 and RMS4 below.

A right and left multiplicative system \mathcal{S} in \mathcal{C} is often called a **multiplicative system** in \mathcal{C} .

MS1 For any $X \in \text{Ob}(\mathcal{C})$, $\text{id}_X \in \mathcal{S}$.

MS2 For any pair (f, g) of morphisms in \mathcal{S} such that the composition $g \circ f$ exists, $g \circ f \in \mathcal{S}$.

LMS3 Any diagram

$$\begin{array}{ccc} & & Y' \\ & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array}$$

in \mathcal{C} with $s \in \mathcal{S}$ can be completed to a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array}$$

in \mathcal{C} with $t \in \mathcal{S}$.

RMS3 Any diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow s & & \\ Y' & & \end{array}$$

in \mathcal{C} with $s \in \mathcal{S}$ can be completed to a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow s & & \downarrow t \\ Y' & \xrightarrow{g} & X' \end{array}$$

in \mathcal{C} with $t \in \mathcal{S}$.

LMS4 Let $f, g : X \rightarrow Y$ be morphisms in \mathcal{C} . If there exists $s : Y \rightarrow Y'$ in \mathcal{S} such that $s \circ f = s \circ g$, then there exists $t : X' \rightarrow X$ in \mathcal{S} such that $f \circ t = g \circ t$.

RMS4 Let $f, g : X \rightarrow Y$ be morphisms in \mathcal{C} . If there exists $t : X' \rightarrow X$ in \mathcal{S} such that $f \circ t = g \circ t$, then there exists $s : Y \rightarrow Y'$ in \mathcal{S} such that $s \circ f = s \circ g$. \square

Definition 3.2. Assume that a family \mathcal{S} of morphisms in \mathcal{C} satisfies MS1 and MS2. Let $X \in \mathcal{C}$.

(i) The category \mathcal{S}^X is defined as follows:

(a) $\text{Ob}(\mathcal{S}^X) := \{s : X \rightarrow X' \mid s \in \mathcal{S}\}$,

(b) $\text{Hom}_{\mathcal{S}^X}((s : X \rightarrow X'), (s' : X \rightarrow X'')) := \{f \in \text{Hom}_{\mathcal{C}}(X', X'') \mid f \circ s = s'\}$.

(ii) Then category \mathcal{S}_X is defined as follows:

(a') $\text{Ob}(\mathcal{S}_X) := \{s : X' \rightarrow X \mid s \in \mathcal{S}\}$,

(b') $\text{Hom}_{\mathcal{S}_X}((s : X' \rightarrow X), (s' : X'' \rightarrow X)) := \{f \in \text{Hom}_{\mathcal{C}}(X', X'') \mid s' \circ f = s\}$. \square

Definition 3.3. A category I is **filtrant** if it satisfies the following three conditions:

(F1) I is nonempty,

(F2) for any i and j in I , there exist $k \in I$ and morphisms $i \rightarrow k, j \rightarrow k$,

(F3) for any morphisms $f, g : i \rightarrow j$, there exists a morphism $h : j \rightarrow k$ such that $h \circ f = h \circ g$. \square

Note that the product category $I \times J$ of filtrant categories I and J is also filtrant.

Proposition 3.4. *Let \mathcal{S} be a family of morphisms in \mathcal{C} .*

(i) *If \mathcal{S} is a left multiplicative system in a category \mathcal{C} , then the category \mathcal{S}_X^{op} is filtrant.*

(ii) *If \mathcal{S} is a right multiplicative system in a category \mathcal{C} , then the category \mathcal{S}^X is filtrant.*

Proof. (i) Let us check the conditions of Definition 3.3.

(F1) Since $(\text{id}_X : X \rightarrow X) \in \mathcal{S}_X$ by MS1, \mathcal{S}_X^{op} is nonempty.

(F2) Let $(s : X' \rightarrow X), (s' : X'' \rightarrow X) \in \mathcal{S}_X$. By LMS3, there exist $t : X''' \rightarrow X$ and $t' : X''' \rightarrow X''$ such that $u : s \circ t = s' \circ t'$ and $t \in \mathcal{S}$. Hence get $(u : X''' \rightarrow X) \in \mathcal{S}$ by MS2.

This is visualized by the diagram

$$\begin{array}{ccc} X''' & \xrightarrow{\quad t \quad} & X' \\ & \searrow u & \downarrow s \\ X'' & \xrightarrow{\quad s' \quad} & X \end{array}$$

Therefore we have the diagram

$$(s : X' \rightarrow X) \xrightarrow{\quad t \quad} (u : X''' \rightarrow X) \xleftarrow{\quad t' \quad} (s' : X'' \rightarrow X)$$

in \mathcal{S}_X^{op} .

(F3) Let $(s : X' \rightarrow X), (s' : X'' \rightarrow X) \in \mathcal{S}_X$ and consider two morphisms $f, g : X'' \rightarrow X'$ with $s \circ f = s \circ g = s'$. By LMS4, there exists $t : W \rightarrow X''$ in \mathcal{S} such that $f \circ t = g \circ t$. Hence $(s' \circ t : W \rightarrow X) \in \mathcal{S}_X^{op}$ and the two compositions

$$(s : X' \rightarrow X) \xrightarrow[\quad g \quad]{\quad f \quad} (s' : X'' \rightarrow X) \xrightarrow{\quad t \quad} (s' \circ t : W \rightarrow X)$$

coincide in \mathcal{S}_X^{op} .

(ii) Similarly, it is easy to check the conditions of Definition 3.3. □

Assume that \mathcal{S} is a left multiplicative system in a category \mathcal{C} . Then, for $X, Y \in \mathcal{C}$, we consider the inductive system in **Set** indexed by \mathcal{S}_X^{op} defined as

$$\mathcal{S}_X^{op} \rightarrow \mathbf{Set}, \quad (s : X' \rightarrow X) \mapsto \text{Hom}_{\mathcal{C}}(X', Y).$$

Each morphism $u_{s',s} : (s : X' \rightarrow X) \rightarrow (s' : X'' \rightarrow X)$ in \mathcal{S}_X^{op} gives a commutative diagram

(3.1)

$$\begin{array}{ccc} X' & \xleftarrow{\quad u_{s',s} \quad} & X'' \\ & \searrow s & \swarrow s' \\ & X & \end{array}$$

Hence we have morphisms

$$\text{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\quad \varphi_{s',s} \quad} \text{Hom}_{\mathcal{C}}(X'', Y)$$

defined by $\varphi_{s',s}(f) = f \circ u_{s',s}$ for $f \in \text{Hom}_{\mathcal{C}}(X', Y)$ and

$$\mathrm{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\varphi_s} \varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{op}} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

Hence an element in $\varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{op}} \mathrm{Hom}_{\mathcal{C}}(X', Y)$ is the equivalence class $[(X', s, f)]$ of a diagram

(X', s, f) in \mathcal{C} of the form

(3.2)

$$\begin{array}{ccc} & X' & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

with $s \in \mathcal{S}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X', Y)$, where the equivalence relation \sim is defined as follows:

$$(X', s, f) \sim (X'', s', f')$$

if and only if there exist morphisms $g : X''' \rightarrow X'$ and $g' : X''' \rightarrow X''$ in \mathcal{C} such that $s \circ g = s' \circ g' \in \mathcal{S}$ and $f \circ g = f' \circ g'$. The diagram of the form (3.2) is often called a *roof*. This is visualized by the commutative diagram

$$\begin{array}{ccccc} & & X' & & \\ & s \swarrow & \uparrow g & \searrow f & \\ X & & X''' & & Y \\ & s'' \swarrow & \downarrow g' & \searrow f' & \\ & & X'' & & \end{array}$$

where $s'' = s \circ g = s' \circ g' \in \mathcal{S}$ and $f'' = f \circ g = f' \circ g'$.

For any morphism $u : X'' \rightarrow X'$ in \mathcal{C} with $s \circ u \in \mathcal{S}$, we have $(X', s, f) \sim (X'', s \circ u, f \circ u)$. This is visualized by the commutative diagram

$$\begin{array}{ccccc} & & X' & & \\ & s \swarrow & \uparrow u & \searrow f & \\ X & & X'' & & Y \\ & s \circ u \swarrow & \parallel & \searrow f \circ u & \\ & & X'' & & \end{array}$$

On the other hand, assume that \mathcal{S} is a right multiplicative system in a category \mathcal{C} . Then, for $X, Y \in \mathcal{C}$, we consider the inductive system in **Set** indexed by \mathcal{S}^Y defined as

$$\mathcal{S}^Y \rightarrow \mathbf{Set}, \quad (t : Y \rightarrow Y') \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y').$$

Each morphism $v_{t',t} : (t : Y \rightarrow Y') \rightarrow (t' : Y \rightarrow Y'')$ in \mathcal{S}^Y gives a commutative diagram

(3.3)

$$\begin{array}{ccc}
 & Y & \\
 t \swarrow & & \searrow t' \\
 Y' & \xrightarrow{v_{t',t}} & Y''
 \end{array}$$

and thus we have a morphism

$$\mathrm{Hom}_{\mathcal{C}}(X, Y') \xrightarrow{\phi_{t',t}} \mathrm{Hom}_{\mathcal{C}}(X, Y'')$$

defined by $\phi_{t',t}(g) = v_{t',t} \circ g$ for $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y')$ and

$$\mathrm{Hom}_{\mathcal{C}}(X, Y') \xrightarrow{\phi_t} \varinjlim_{(Y \rightarrow Y') \in \mathcal{S}^Y} \mathrm{Hom}_{\mathcal{C}}(X, Y').$$

Hence an element in $\varinjlim_{(Y \rightarrow Y') \in \mathcal{S}^Y} \mathrm{Hom}_{\mathcal{C}}(X, Y')$ is the equivalence class $[(Y', f, t)]$ of a diagram

(Y', f, t) in \mathcal{C} of the form

(3.4)

$$\begin{array}{ccc}
 X & & Y \\
 & \searrow f & \swarrow t \\
 & Y' &
 \end{array}$$

with $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y')$ and $t \in \mathcal{S}$, where the equivalence relation \sim is defined as follows:

$$(Y', f, t) \sim (Y'', f', t')$$

if and only if there exist morphisms $h : Y' \rightarrow Y''$ and $h' : Y'' \rightarrow Y'''$ in \mathcal{C} such that $h \circ t = h' \circ t' \in \mathcal{S}$ and $h \circ f = h' \circ f'$. This is visualized by the commutative diagram

$$\begin{array}{ccccc}
 & & Y' & & \\
 & f \nearrow & \downarrow h & \nwarrow t & \\
 X & \xrightarrow{f''} & Y''' & \xleftarrow{t''} & Y \\
 & f' \searrow & \uparrow h' & \swarrow t' & \\
 & & X'' & &
 \end{array}$$

where $f'' = h \circ f = h' \circ f'$ and $t'' = h \circ t = h' \circ t' \in \mathcal{S}$.

For any morphism $v : Y' \rightarrow Y''$ in \mathcal{C} with $v \circ t \in \mathcal{S}$, we have $(Y', f, t) \sim (Y'', v \circ f, v \circ t)$. This is visualized by the commutative diagram

$$\begin{array}{ccccc}
 & & Y'' & & \\
 & v \circ f \nearrow & \parallel & \nwarrow v \circ t & \\
 X & & Y'' & & Y \\
 & f \searrow & \uparrow v & \swarrow t & \\
 & & Y' & &
 \end{array}$$

Now assume that \mathcal{S} is a right and left multiplicative system in a category \mathcal{C} . For $X, Y \in \mathcal{C}$, we can consider the inductive system in **Set** indexed by $\mathcal{S}_X^{op} \times \mathcal{S}^Y$ defined as

$$\mathcal{S}_X^{op} \times \mathcal{S}^Y \Rightarrow \mathbf{Set}, \quad (s : X' \rightarrow X, t : Y \rightarrow Y') \mapsto \text{Hom}_{\mathcal{C}}(X', Y').$$

Each morphism

$$w_{(s',t'),(s,t)} = (u_{s',s}, v_{t',t}) : (s : X' \rightarrow X, t : Y \rightarrow Y') \rightarrow (s' : X'' \rightarrow X, t' : Y \rightarrow Y'')$$

in $\mathcal{S}_X^{op} \times \mathcal{S}^Y$ gives diagrams (3.3) and (3.1) as above. Hence we have morphisms

$$\text{Hom}_{\mathcal{C}}(X', Y') \xrightarrow{\rho_{(s',t'),(s,t)}} \text{Hom}_{\mathcal{C}}(X'', Y'')$$

defined by $\rho_{(s',t'),(s,t)}(f) = v_{t',t} \circ f \circ u_{s',s}$ for $f \in \text{Hom}_{\mathcal{C}}(X', Y')$ and

$$\text{Hom}_{\mathcal{C}}(X', Y') \xrightarrow{\rho_{(s,t)}} \varinjlim_{(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{op} \times \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X', Y').$$

Hence an element in $\varinjlim_{(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{op} \times \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X', Y')$ is the equivalence class $[(X', Y', s, f, t)]$

$$(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{op} \times \mathcal{S}^Y$$

of a diagram (X', Y', s, f, t) in \mathcal{C} of the form

(3.5)

$$\begin{array}{ccccc} & & X' & & Y \\ & \swarrow s & & \searrow f & \swarrow t \\ X & & & & Y' \end{array}$$

with $f \in \text{Hom}_{\mathcal{C}}(X', Y')$ and $s, t \in \mathcal{S}$, where the equivalence relation \sim is defined as follows:

$$(X', Y', s, f, t) \sim (X'', Y'', s', f', t')$$

if and only if there exist morphisms $g : X''' \rightarrow X', g' : X''' \rightarrow X'', h : Y' \rightarrow Y''$ and $h' : Y'' \rightarrow Y'''$ in \mathcal{C} such that $h \circ f \circ g = h' \circ f' \circ g', s \circ g = s' \circ g' \in \mathcal{S}$ and $h \circ t = h' \circ t' \in \mathcal{S}$. This is visualized by the commutative diagram

$$\begin{array}{ccccccc} & & X' & \xrightarrow{f} & Y' & & \\ & \swarrow s & \uparrow g & & \downarrow h & \swarrow t & \\ X & \xleftarrow{s''} & X''' & \xrightarrow{f''} & Y''' & \xleftarrow{t''} & Y \\ & \swarrow s' & \downarrow g' & & \uparrow h' & \swarrow t' & \\ & & X'' & \xrightarrow{f'} & Y'' & & \end{array}$$

where $f'' = h \circ f \circ g = h' \circ f' \circ g', s'' = s \circ g = s' \circ g' \in \mathcal{S}$ and $t'' = h \circ t = h' \circ t' \in \mathcal{S}$.

Proposition 3.5. *Let \mathcal{S} be a multiplicative system in a category \mathcal{C} . Then we have*

$$\varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{op}} \text{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{(i)} \varinjlim_{(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{op} \times \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X', Y') \xrightarrow{(ii)} \varinjlim_{(Y \rightarrow Y') \in \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X, Y').$$

Proof. (i) From the diagram

$$\varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{op}} \text{Hom}_{\mathcal{C}}(X', Y) \xleftarrow{\varphi_s} \text{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\rho_{(s, \text{id}_Y)}} \varinjlim_{(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{op} \times \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X', Y),$$

we have a morphism

$$\lim_{\substack{\longrightarrow \\ (X' \rightarrow X) \in \mathcal{S}_X^{op}}} \text{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\phi} \lim_{\substack{\longrightarrow \\ (X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{op} \times \mathcal{S}^Y}} \text{Hom}_{\mathcal{C}}(X', Y'),$$

where ϕ sends $[(X', s, f)]$ to $[(X', Y, s, f, \text{id}_Y)]$.

Let $[(X', s, f)]$ and $[(X'', s', f')]$ be any elements of $\lim_{\substack{\longrightarrow \\ (X' \rightarrow X) \in \mathcal{S}_X^{op}}} \text{Hom}_{\mathcal{C}}(X', Y)$.

Assume that $\phi([(X', s, f)]) = \phi([(X'', s', f')])$. Then there exist morphisms $g : X''' \rightarrow X'$, $g' : X''' \rightarrow X''$ and $h : Y \rightarrow Y'''$ in \mathcal{C} such that $s \circ g = s' \circ g' \in \mathcal{S}$, $h \in \mathcal{S}$ and $h \circ f \circ g = h \circ f' \circ g'$. This is visualized by the commutative diagram

$$\begin{array}{ccccc} & & X' & \xrightarrow{f} & Y \\ & s \swarrow & \uparrow g & & \nwarrow \text{id}_Y \\ X & \xleftarrow{s''} & X''' & \xrightarrow{f''} & Y''' & \xleftarrow{t''} & Y \\ & \swarrow s' & \downarrow g' & & \uparrow h' & \nwarrow \text{id}_Y \\ & & X'' & \xrightarrow{f'} & Y \end{array}$$

By LMS4, there exists a morphism $(v : V \rightarrow X''') \in \mathcal{S}$ such that $s \circ g \circ v = s' \circ g' \circ v \in \mathcal{S}$ and $f \circ g \circ v = f' \circ g' \circ v$. Hence we obtain $[(X', s, f)] = [(V, s \circ g \circ v, f \circ g \circ v)] = [(X'', s', f')]$, and thus the morphism ϕ is injective.

Let $[(X', Y', s, f, t)]$ be any element in $\lim_{\substack{\longrightarrow \\ (X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{op} \times \mathcal{S}^Y}} \text{Hom}_{\mathcal{C}}(X', Y')$, which is represented by a diagram (X', Y', s, f, t) in \mathcal{C} of the form

(3.6)

$$\begin{array}{ccccc} & & X' & & Y \\ & s \swarrow & & \searrow f & \nwarrow t \\ & X & & Y' & \end{array}$$

with $f \in \text{Hom}_{\mathcal{C}}(X', Y')$ and $s, t \in \mathcal{S}$. By LMS3, there exist morphisms $f' : X'' \rightarrow Y$ and $t' : X'' \rightarrow X'$ such that $t \circ f' = f \circ t'$ and $t' \in \mathcal{S}$. Then we have the following commutative diagram

$$\begin{array}{ccccc} & & X' & \xrightarrow{f} & Y' \\ & s \swarrow & \uparrow t' & & \nwarrow t \\ X & \xleftarrow{s''} & X''' & \xrightarrow{f''} & Y''' & \xleftarrow{t''} & Y \\ & \swarrow s \circ t' & \parallel & & \uparrow t & \nwarrow \parallel \\ & & X'' & \xrightarrow{f'} & Y \end{array}$$

Therefore $\varphi([(X'', s \circ t', f')]) = [(X', Y', s, f, t)]$, and thus the morphism φ is surjective.

(ii) From the diagram

$$\varinjlim_{(Y \rightarrow Y') \in \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X, Y') \xleftarrow{\phi_t} \text{Hom}_{\mathcal{C}}(X, Y') \xrightarrow{\rho(\text{id}_X, t)} \varinjlim_{(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{\text{op}} \times \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X', Y'),$$

we have a morphism

$$\varinjlim_{(Y \rightarrow Y') \in \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X, Y') \xrightarrow{\phi} \varinjlim_{(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{\text{op}} \times \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X', Y'),$$

where ϕ sends $[(Y', f, t)]$ to $[(X, Y', \text{id}_X, f, t)]$.

Let $[(Y', f, t)]$ and $[(Y'', f', t')]$ be any elements of $\varinjlim_{(Y \rightarrow Y') \in \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X, Y')$. Assume

that $\phi([(Y', f, t)]) = \phi([(Y'', f', t')])$. Then there exist morphisms $g : X''' \rightarrow X, h : Y' \rightarrow Y'''$ and $h' : Y'' \rightarrow Y'''$ in \mathcal{C} such that $g \in \mathcal{S}, h \circ t = h' \circ t' \in \mathcal{S}$ and $h \circ f \circ g = h' \circ f' \circ g$.

This is visualized by the commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y' \\ & \swarrow \text{id}_X & \uparrow g & & \downarrow h \\ X & & X''' & \xrightarrow{f''} & Y''' \\ & \searrow \text{id}_X & \downarrow g & & \uparrow h' \\ & & X & \xrightarrow{f'} & Y'' \end{array}$$

By RMS4, there exists a morphism $(u : Y''' \rightarrow U) \in \mathcal{S}$ such that $u \circ h \circ f = u \circ h' \circ f'$ and $u \circ h \circ t = u \circ h' \circ t' \in \mathcal{S}$. Hence we obtain $[(Y', f, t)] = [(U, u \circ h \circ f, u \circ h \circ t)] = [(Y'', f', t')]$, and thus the morphism φ is injective.

Let $[(X', Y', s, f, t)]$ be any element in $\varinjlim_{(X' \rightarrow X, Y \rightarrow Y') \in \mathcal{S}_X^{\text{op}} \times \mathcal{S}^Y} \text{Hom}_{\mathcal{C}}(X', Y')$, which is represented by a diagram (X', Y', s, f, t) in \mathcal{C} of the form

(3.7)

$$\begin{array}{ccccc} & & X' & & Y \\ & \swarrow s & & \searrow f & \swarrow t \\ X & & & & Y' \end{array}$$

with $f \in \text{Hom}_{\mathcal{C}}(X', Y')$ and $s, t \in \mathcal{S}$. By RMS3, there exist morphisms $f' : X \rightarrow Y''$ and $s' : Y' \rightarrow Y''$ such that $s' \circ f = f' \circ s$ and $s' \in \mathcal{S}$. Then we have the following commutative diagram

$$\begin{array}{ccccc}
 & X' & \xrightarrow{f} & Y' & \\
 & \parallel & & \downarrow s' & \nearrow t \\
 X & \xleftarrow{s} & X' & \xrightarrow{f''} & Y'' \\
 & \parallel & & \parallel & \nwarrow s' \circ t \\
 & X & \xrightarrow{f'} & Y'' &
 \end{array}$$

Hence $\phi([(Y'', f', s' \circ t)]) = [(X', Y', s, f, t)]$, and thus the morphism ϕ is surjective. \square

Corollary 3.6. *Let S be a multiplicative system in a category \mathcal{C} . Then the isomorphism*

$$\lim_{\substack{\longrightarrow \\ (Y \rightarrow Y') \in S^Y}} \text{Hom}_{\mathcal{C}}(X, Y') \simeq \lim_{\substack{\longrightarrow \\ (X' \rightarrow X) \in S_X^{op}}} \text{Hom}_{\mathcal{C}}(X', Y)$$

is given by

$$[(X', s, f)] \longleftrightarrow [(Y', g, t)],$$

where (X', s, f) and (Y', g, t) are completed each other by RMS3 and LMS3 as follows:

$$\begin{array}{ccc}
 X' & \xrightarrow{f} & Y \\
 s \downarrow & & \downarrow t \\
 X & \xrightarrow{g} & Y'
 \end{array}$$

Proof. This follows from the proof of Proposition 3.5. \square

4 Construction of Localization

Definition 4.1. Let S be a left multiplicative system in a category \mathcal{C} .

(i) We set $\text{Ob}(\mathcal{C}_S) := \text{Ob}(\mathcal{C})$.

(ii) We set

$$\text{Hom}_{\mathcal{C}_S}(X, Y) := \lim_{\substack{\longrightarrow \\ (X' \rightarrow X) \in S_X^{op}}} \text{Hom}_{\mathcal{C}}(X', Y)$$

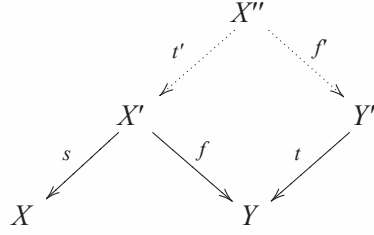
for each ordered pair of objects $X, Y \in \text{Ob}(\mathcal{C})$. \square

Let us show that \mathcal{C}_S is a category.

Lemma 4.2. *Assume that S be a left multiplicative system in a category \mathcal{C} . Let $X \in \mathcal{C}$ and $(t : Y' \rightarrow Y) \in S$. Then t induces a bijection*

$$t_* : \text{Hom}_{\mathcal{C}_S}(X, Y') \xrightarrow{\sim} \text{Hom}_{\mathcal{C}_S}(X, Y), \quad [(X', s, f)] \mapsto [(X', s, t \circ f)].$$

Proof. By LMS3, we have the following diagram

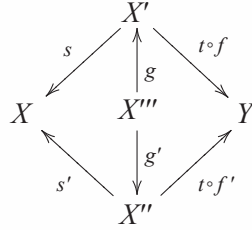


with $t' \in \mathcal{S}$. Then

$$t_*([X'', s \circ t', f']) = [(X'', s \circ t', t \circ f')] = [(X'', s \circ t', f \circ t')] = [(X', s, f)].$$

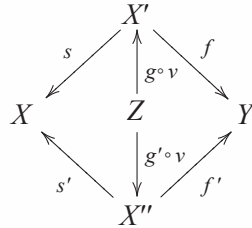
Hence the map t_* is surjective.

Let $[(X', s, f)], [(X'', s', f')] \in \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y')$ such that $[(X', s, t \circ f)] = [(X'', t \circ f', s')] \in \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y)$. There exists the following commutative diagram in \mathcal{C} :



with $s \circ g \in \mathcal{S}$. Since $t \circ (f \circ g) = t \circ (f' \circ g')$, by LMS4, there exists a morphism $v : Z \rightarrow X''' \in \mathcal{S}$

such that $(f \circ g) \circ v = (f' \circ g') \circ v$. Then we obtain the following commutative diagram in \mathcal{C} :



with $s \circ g \circ v \in \mathcal{S}$. Hence we get $[(X', s, f)] = [(X'', s', f')]$, and thus the map t_* is injective. \square

Assume that \mathcal{S} be a left multiplicative system in a category \mathcal{C} . Then by using Lemma 4.2 and the composition map in \mathcal{C} , we define the composition map in $\mathcal{C}_{\mathcal{S}}$

$$(4.1) \quad \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y) \times \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Z)$$

as follows:

$$\begin{aligned}
 \text{Hom}_{\mathcal{C}_S}(X, Y) \times \text{Hom}_{\mathcal{C}_S}(Y, Z) &= \text{Hom}_{\mathcal{C}_S}(X, Y) \times \varinjlim_{(Y' \rightarrow Y) \in \mathcal{S}_Y^{op}} \text{Hom}_{\mathcal{C}}(Y', Z) \\
 &\simeq \varinjlim_{(Y' \rightarrow Y) \in \mathcal{S}_Y^{op}} \left(\text{Hom}_{\mathcal{C}_S}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y', Z) \right) \\
 &\rightarrow \varinjlim_{(Y' \rightarrow Y) \in \mathcal{S}_Y^{op}} \left(\text{Hom}_{\mathcal{C}_S}(X, Y') \times \text{Hom}_{\mathcal{C}}(Y', Z) \right) \\
 &= \varinjlim_{(Y' \rightarrow Y) \in \mathcal{S}_Y^{op}} \left(\left(\varinjlim_{(Y' \rightarrow X) \in \mathcal{S}_X^{op}} \text{Hom}_{\mathcal{C}}(X', Y') \right) \times \text{Hom}_{\mathcal{C}}(Y', Z) \right) \\
 &\simeq \varinjlim_{(Y' \rightarrow Y) \in \mathcal{S}_Y^{op}} \varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{op}} \left(\text{Hom}_{\mathcal{C}}(X', Y') \times \text{Hom}_{\mathcal{C}}(Y', Z) \right) \\
 &\rightarrow \varinjlim_{(Y' \rightarrow Y) \in \mathcal{S}_Y^{op}} \varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{op}} \text{Hom}_{\mathcal{C}}(X', Z) \\
 &\simeq \varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{op}} \text{Hom}_{\mathcal{C}}(X', Z) \\
 &= \text{Hom}_{\mathcal{C}_S}(X, Z).
 \end{aligned}$$

The composition in \mathcal{C}_S is described by roofs as follows:

For $\alpha = [(X', s, f)] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$ and $\beta = [(Y', t, g)] \in \text{Hom}_{\mathcal{C}_S}(Y, Z)$, by definition the composition map is given by the following commutative diagram

$$\begin{array}{ccccc}
 & & X'' & & \\
 & \swarrow t' & & \searrow f' & \\
 & X' & & Y' & \\
 \swarrow s & & \searrow f & \swarrow t & \searrow g \\
 X & & Y & & Z
 \end{array}$$

with $t' \in \mathcal{S}$, and thus we obtain

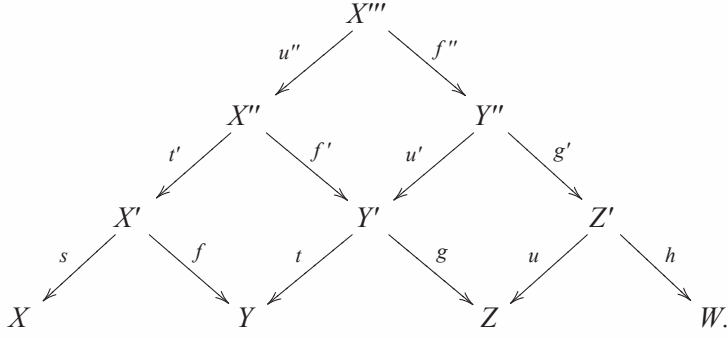
$$\beta \circ \alpha = [(Y', t, g)] \circ [(X', s, f)] = [(X'', s \circ t', g \circ f')].$$

Proposition 4.3. *The composition (4.1) is associative.*

Proof. Let $\alpha \in \text{Hom}_{\mathcal{C}_S}(X, Y)$, $\beta \in \text{Hom}_{\mathcal{C}_S}(Y, Z)$ and $\gamma \in \text{Hom}_{\mathcal{C}_S}(Z, W)$, and assume that $\alpha = [(X', s, f)]$, $\beta = [(Y', t, g)]$ and $\gamma = [(Z', u, h)]$:

$$\begin{array}{ccc}
 \begin{array}{ccc} & X' & \\ \swarrow s & & \searrow f \\ X & & Y \end{array} & \begin{array}{ccc} & Y' & \\ \swarrow t & & \searrow g \\ Y & & Z \end{array} & \begin{array}{ccc} & Z' & \\ \swarrow u & & \searrow h \\ Z & & W \end{array}
 \end{array}$$

Then using the axiom LMS3, the composition is obtained from the commutative diagram



Then we get

$$\begin{aligned}
 (\gamma \circ \beta) \circ a &= ([Z', u, h] \circ [Y', t, g]) \circ [X', s, f] \\
 &= [Y'', t \circ u', h \circ g'] \circ [X', s, f] \\
 &= [X''', s \circ (t' \circ u''), (h \circ g') \circ f''] \\
 &= [X''', (s \circ t') \circ u'', h \circ (g' \circ f'')] \\
 &= [Z', u, h] \circ [X'', s \circ t', g \circ f'] \\
 &= [Z', u, h] \circ ([Y', t, g] \circ [X', s, f]) \\
 &= \gamma \circ (\beta \circ a).
 \end{aligned}$$

This completes the proof. □

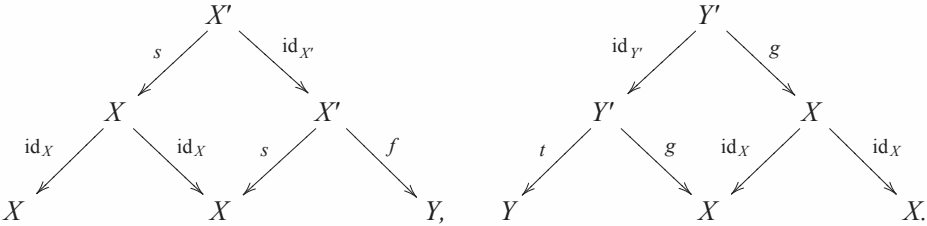
For any $X \in \text{Ob}(\mathcal{C}_S)$,

$$[(X, \text{id}_X, \text{id}_X)] \in \text{Hom}_{\mathcal{C}_S}(X, X)$$

is the **identity morphism**. Indeed, we have

$$[(X', s, f)] \circ [(X, \text{id}_X, \text{id}_X)] = [(X', s, f)] \text{ and } [(X, \text{id}_X, \text{id}_X)] \circ [(Y', t, g)] = [(Y', t, g)]$$

for any $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and any $g \in \text{Hom}_{\mathcal{C}}(Y, X)$. This is visualized by the commutative diagrams



Therefore, for a left multiplicative system \mathcal{S} in a category \mathcal{C} , we obtain a category \mathcal{C}_S constructed as above.

Definition 4.4. Assume that \mathcal{S} be a left multiplicative system in a category \mathcal{C} . Then we define the natural functor $Q_S : \mathcal{C} \rightarrow \mathcal{C}_S$ as follows:

- (i) $Q_S(X) := X$ for each $X \in \mathcal{C}$, and
- (ii) $Q_S(f) := [(X, \text{id}_X, f)]$ for each $f \in \text{Hom}_{\mathcal{C}}(X, Y)$.

If there is no risk of confusion, we denote this functor simply by Q . □

Lemma 4.5. *The functor Q has the following properties.*

- (i) *If $(s : X \rightarrowtail Y) \in \mathcal{S}$, then $Q(s)$ is an isomorphism and $Q(s)^{-1} = [(X, s, \text{id}_X)]$.*
- (ii) *For $\alpha = [(X', s, f)] \in \text{Hom}_{\mathcal{C}}(X, Y)$, we have $\alpha = Q(f) \circ Q(s)^{-1}$. Moreover, if $G : \mathcal{C}_{\mathcal{S}} \rightarrow \mathcal{D}$ is any functor, then for $(s : X' \rightarrowtail X) \in \mathcal{S}$, we have $G(\alpha) = G([(X', s, f)]) = G(Q(f)) \circ G(Q(s))^{-1} \in \text{Hom}_{\mathcal{D}}(G(X), G(Y))$.*
- (iii) *For two morphisms $f, g : X \rightarrow Y$ in \mathcal{C} , $Q(f) = Q(g)$ in $\mathcal{C}_{\mathcal{S}}$ if and only if there exists a morphism $s : X' \rightarrowtail X$ in \mathcal{S} such that $f \circ s = g \circ s$.*
- (iv) *Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . If $Q(f)$ is an isomorphism, then there exists a morphism $g : W \rightarrowtail X$ in \mathcal{C} such that $f \circ g \in \mathcal{S}$.*

Proof. (i) Since

$$[(X, \text{id}_X, s)] \circ [(X, s, \text{id}_X)] = [(X, s, s)] = [(Y, \text{id}_Y, \text{id}_Y)]$$

and

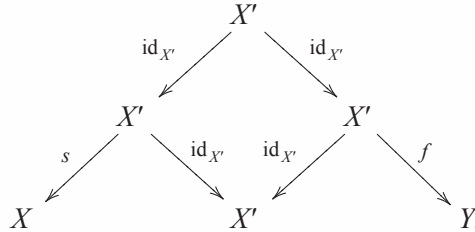
$$[(X, s, \text{id}_X)] \circ [(X, \text{id}_X, s)] = [(X, \text{id}_X, \text{id}_X)].$$

$Q(s) = [(X, \text{id}_X, s)]$ has a right inverse and a left inverse. Hence $Q(s)$ is an isomorphism and $Q(s)^{-1} = [(X, s, \text{id}_X)]$.

(ii) We have

$$\alpha = [(X', s, f)] = [(X', \text{id}_{X'}, f)] \circ [(X', s, \text{id}_{X'})] = Q(f) \circ Q(s)^{-1}.$$

This is visualized by the commutative diagram



Moreover, for $(s : X' \rightarrowtail X) \in \mathcal{S}$, we have

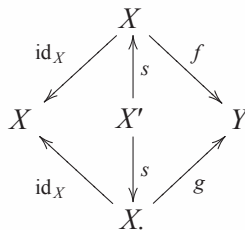
$$G(Q(s)) \circ G(Q(s)^{-1}) = G(Q(s) \circ Q(s)^{-1}) = G(\text{id}_X) = \text{id}_{G(X)},$$

$$G(Q(s)^{-1}) \circ G(Q(s)) = G(Q(s)^{-1} \circ Q(s)) = G(\text{id}_{X'}) = \text{id}_{G(X')}.$$

Hence $G(Q(s))$ is also an isomorphism and $G(Q(s))^{-1} = G(Q(s)^{-1}) \in \text{Hom}_{\mathcal{D}}(G(X), G(X'))$. Therefore we obtain

$$G(\alpha) = G([(X', s, f)]) = G(Q(f)) \circ G(Q(s))^{-1} \in \text{Hom}_{\mathcal{D}}(G(X), G(X')).$$

(iii) This follows from the diagram



(iv) If $Q(f)^{-1} := [(X', s, f')]$ in \mathcal{C}_S , then we get $Q(f \circ f') = Q(f) \circ Q(f') = Q(s)$ from Lemma

4.5.(ii). By Lemma 4.5.(iii), there exists $t : W \rightarrow X'$ in \mathcal{S} such that $f \circ f' \circ t = s \circ t$. Put $g := f' \circ t$. Then we obtain $g : W \rightarrow X$ and $f \circ g \in \mathcal{S}$ by MS2. \square

Theorem 4.6. *Let \mathcal{C} be a category and let \mathcal{S} be a left multiplicative system in \mathcal{C} . Then the category \mathcal{C}_S and the functor Q_S define a localization of \mathcal{C} by \mathcal{S} .*

Proof. Let us check the conditions of Definition 2.1.

(L1) This follows from Lemma 4.5.(i).

(L2) For $X \in \text{Ob}(\mathcal{C}_S) = \text{Ob}(\mathcal{C})$, set $F_S(X) := F(X)$. For $X, Y \in \mathcal{C}$, we have

$$\begin{aligned} \text{Hom}_{\mathcal{C}_S}(X, Y) &= \varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{\text{op}}} \text{Hom}_{\mathcal{C}}(X', Y) \\ &\rightarrow \varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{\text{op}}} \text{Hom}_{\mathcal{D}}(F(X'), F(Y)) \\ &\simeq \varinjlim_{(X' \rightarrow X) \in \mathcal{S}_X^{\text{op}}} \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ &\simeq \text{Hom}_{\mathcal{D}}(F_S(X), F_S(Y)). \end{aligned}$$

Hence, for $[(X', s, f)] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$, we obtain

$$F_S([(X', s, f)]) = [(F(X'), F(s), F(f))] = [(F(X'), \text{id}_{F(X)}, F(f) \circ F(s)^{-1})] = F(f) \circ F(s)^{-1}.$$

For $[(X, \text{id}_X, \text{id}_X)] \in \text{Hom}_{\mathcal{C}_S}(X, X)$, we obtain

$$F_S([(X, \text{id}_X, \text{id}_X)]) = [(F(X), F(\text{id}_X), F(\text{id}_X))] = [(F(X), \text{id}_{F(X)}, \text{id}_{F(X)})] = \text{id}_{F_S(X)}.$$

Let $[(X', s, f)] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$ and $[(Y', t, g)] \in \text{Hom}_{\mathcal{C}_S}(Y, Z)$. Then we get

$$\begin{aligned} F_S([(Y', t, g)] \circ [(X', s, f)]) &= F_S([(X'', s \circ u, g \circ h)]) \\ &= [(F(X''), \text{id}_{F(X)}, F(g \circ h) \circ F(s \circ u)^{-1})] \\ &= [(F(X''), \text{id}_{F(X)}, F(g) \circ F(h) \circ F(u)^{-1} \circ F(s)^{-1})] \\ &= F(g) \circ F(h) \circ F(u)^{-1} \circ F(s)^{-1}. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} F_S([(Y', t, g)]) \circ F_S([(X', s, f)]) &= F(g) \circ F(t)^{-1} \circ F(f) \circ F(s)^{-1} \\ &= F(g) \circ F(t)^{-1} \circ F(f) \circ F(s)^{-1} \\ &= F(g) \circ F(h) \circ F(u)^{-1} \circ F(s)^{-1}. \end{aligned}$$

This is visualized by the commutative diagram

$$\begin{array}{ccccc} & & F(X'') & & \\ & \nearrow F(u) & & \searrow F(h) & \\ & F(X') & & F(Y') & \\ & \nwarrow F(s) & \searrow F(f) & \nwarrow F(t) & \searrow F(g) \\ F(X) & & F(Y) & & F(Z) \end{array}$$

$\begin{array}{ccc} \text{Solid arrows: } F(u), F(h), F(s), F(f), F(t), F(g) \\ \text{Dashed arrows: } F(u)^{-1}, F(s)^{-1}, F(t)^{-1} \end{array}$

with $u \in \mathcal{S}$. Hence we obtain

$$F_S([(Y', t, g)]) \circ [(X', s, f)] = F_S([(Y', t, g)]) \circ F_S([(X', s, f)]).$$

Therefore $F_S : \mathcal{C}_S \rightarrow \mathcal{D}$ is the functor which we required.

Finally, let $X \in \mathcal{C}$ and let $f : X \rightarrow Y$ be any morphism in \mathcal{C} . Then we have

$$(F_S \circ Q_S)(X) = F_S(Q_S(X)) = F_S(X) = F(X)$$

and

$$(F_S \circ Q_S)(f) = F_S(Q_S(f)) = F_S([(X, \text{id}_X, f)]) = F(f) \circ F(\text{id}_X)^{-1} = F(f).$$

Hence we obtain $F = F_S \circ Q_S$.

(L3) Let G_1, G_2 be two objects of $\text{Fct}(\mathcal{C}_S, \mathcal{D})$. By definition, a morphism of functors $\theta : G_1 \rightarrow G_2$ consists of a morphism $\theta_X : G_1(X) \rightarrow G_2(X)$ for all $X \in \mathcal{C}_S$ such that for all $\alpha \in \text{Hom}_{\mathcal{C}_S}(X, Y)$, the following diagram commutes:

(4.2)

$$\begin{array}{ccc} G_1(X) & \xrightarrow{\theta_X} & G_2(X) \\ G_1(\alpha) \downarrow & & \downarrow G_2(\alpha) \\ G_1(Y) & \xrightarrow{\theta_Y} & G_2(Y). \end{array}$$

Since $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}_S)$, we have $G_1(X) = G_1(Q(X))$ and $G_2(X) = G_2(Q(X))$ for all $X \in \mathcal{C}$, and by Lemme 4.5(ii), we can rewrite the diagram (4.2) as follows:

$$\begin{array}{ccc} G_1(Q(X)) & \xrightarrow{Q^*(\theta)_X} & G_2(Q(X)) \\ G_1(Q(s))^{-1} \downarrow & & \downarrow G_2(Q(s))^{-1} \\ G_1(Q(X')) & \xrightarrow{Q^*(\theta)_{X'}} & G_2(Q(X')) \\ G_1(Q(f)) \downarrow & & \downarrow G_2(Q(f)) \\ G_1(Q(Y)) & \xrightarrow{Q^*(\theta)_Y} & G_2(Q(Y)). \end{array}$$

in \mathcal{D} for $\alpha = [(X', s, f)] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$.

(i) Let $\theta_1, \theta_2 : G_1 \rightarrow G_2$ be two morphisms of functors. Assume that $Q^*(\theta_1) = Q^*(\theta_2)$. Then we have $(\theta_1)_X = Q^*(\theta_1)_X = Q^*(\theta_2)_X = (\theta_2)_X$ for all $X \in \mathcal{C}_S$. Hence $\theta_1 = \theta_2$, and thus the map Q^* is injective.

(ii) Suppose that a morphism of functors $\lambda : G_1 \circ Q \rightarrow G_2 \circ Q$ is given. Then we set $\theta_X := \lambda_X$, and by Lemma 4.5(ii), we obtain a commutative diagram

$$\begin{array}{ccccc}
 G_1(X) & \xrightarrow{\theta_X} & G_2(X) & & \\
 \downarrow G_1(\alpha) & \searrow & \downarrow G_2(\alpha) & & \\
 & G_1(Q(X)) \xrightarrow{\lambda_X} G_2(Q(X)) & & & \\
 & \uparrow G_1(Q(s)) \quad \uparrow G_2(Q(s)) & & & \\
 & G_1(Q(X')) \xrightarrow{\lambda_{X'}} G_2(Q(X')) & & & \\
 & \downarrow G_1(Q(f)) \quad \downarrow G_2(Q(f)) & & & \\
 & G_1(Q(Y)) \xrightarrow{\lambda_Y} G_2(Q(Y)) & & & \\
 \downarrow & \swarrow & \downarrow & & \\
 G_1(Y) & \xrightarrow{\theta_Y} & G_2(Y) & &
 \end{array}$$

in \mathcal{D} for each $\alpha = [(X', s, f)] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$. Hence we get a morphism of functors $\theta : G_1 \rightarrow G_2$. Since $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}_S)$, and $G_1(Q(\text{id}_X)) = \text{id}_{G_1(Q(X))}$ and $G_2(Q(\text{id}_X)) = \text{id}_{G_2(Q(X))}$, we have $Q^*(\theta) = \lambda$. Hence the map Q^* is surjective. \square

Remark 4.7. Let S be a right multiplicative system in a category \mathcal{C} . In the same way, we set

(i) $\text{Ob}(\mathcal{C}_S^r) := \text{Ob}(\mathcal{C})$, and

(ii) for each ordered pair of objects $X, Y \in \text{Ob}(\mathcal{C})$,

$$\text{Hom}_{\mathcal{C}_S^r}(X, Y) := \varinjlim_{(Y \rightarrow Y') \in S^Y} \text{Hom}_{\mathcal{C}}(X, Y').$$

Then we obtain a category \mathcal{C}_S^r . Moreover, we define the natural functor $Q_S^r : \mathcal{C} \rightarrow \mathcal{C}_S^r$ as

(i) $Q_S^r(X) := X$ for each $X \in \mathcal{C}$, and

(ii) $Q_S^r(f) := [(Y, f, \text{id}_Y)]$ for each $f \in \text{Hom}_{\mathcal{C}}(X, Y)$.

Then we obtain a localization (\mathcal{C}_S^r, Q_S^r) of \mathcal{C} by S . If S is a multiplicative system in a category \mathcal{C} , then the categories \mathcal{C}_S and \mathcal{C}_S^r are equivalent by Proposition 3.5. \square

Proposition 4.8. Let \mathcal{C} be a category and let S be a left multiplicative system in \mathcal{C} . Consider morphisms $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ in \mathcal{C} and morphisms $\alpha : X \rightarrow X'$ and $\beta : Y \rightarrow Y'$ in \mathcal{C}_S . Assume that $Q(f') \circ \alpha = \beta \circ Q(f)$. Then there exist a morphism $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ in \mathcal{C} and a commutative diagram in \mathcal{C}

$$\begin{array}{ccccc}
 X & \xleftarrow{s} & \tilde{X} & \xrightarrow{u} & X' \\
 f \downarrow & & \downarrow \tilde{f} & & \downarrow f' \\
 Y & \xleftarrow{t} & \tilde{Y} & \xrightarrow{v} & Y'
 \end{array}$$

with $s, t \in \mathcal{S}$, $\alpha = Q(u) \circ Q(s)^{-1}$ and $\beta = Q(v) \circ Q(t)^{-1}$.

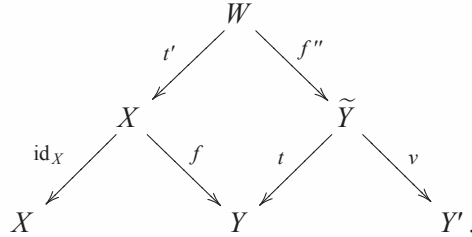
Proof. Put $\alpha = [(A, r, a)]$ and $\beta = [(\tilde{Y}, t, v)]$. Then we have

$$Q(f') \circ \alpha = [(X', \text{id}_{X'}, f')] \circ [(A, r, a)] = [(A, r, f' \circ a)]$$

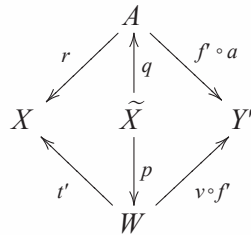
and

$$\beta \circ Q(f) = [(\tilde{Y}, t, v)] \circ [(X, \text{id}_X, f)] = [(W, t', v \circ f'')].$$

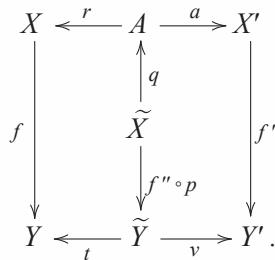
This is visualized by the commutative diagram



Since $Q(f') \circ \alpha = \beta \circ Q(f)$, we have an object $\tilde{X} \in \mathcal{C}$, morphisms $p : \tilde{X} \rightarrow W$ and $q : \tilde{X} \rightarrow A$, and the commutative diagram



with $t' \circ p = r \circ q \in \mathcal{S}$. Hence we obtain the commutative diagram



We set $\tilde{f} := f'' \circ p$, $u := a \circ q$, and $s := r \circ q \in \mathcal{S}$. Then

$$\alpha = [(A, r, a)] = [(\tilde{X}, r \circ q, a \circ q)] = [(\tilde{X}, s, u)] = Q(u) \circ Q(s)^{-1}.$$

This completes the proof. □

5 Saturated multiplicative systems

Definition 5.1. Let \mathcal{C} be a category and let \mathcal{S} be a family of morphisms in \mathcal{C} .

- (i) Assume that \mathcal{S} is a left multiplicative system. \mathcal{S} is said to be **left saturated** if it satisfies the following axiom.

LMS5 for any morphism $f: X \rightarrow Y, g: W \rightarrow X$ and $h: V \rightarrow W$ such that $f \circ g$ and $g \circ h$ belong to \mathcal{S} , the morphism f belongs to \mathcal{S} .

(ii) Assume that \mathcal{S} is a right multiplicative system. \mathcal{S} is said to be **right saturated** if it satisfies the following axiom.

RMS5 for any morphism $f: X \rightarrow Y, g: W \rightarrow X$ and $h: V \rightarrow W$ such that $f \circ g$ and $g \circ h$ belong to \mathcal{S} , the morphism h belongs to \mathcal{S} .

(iii) Assume that \mathcal{S} is a right and left multiplicative system. \mathcal{S} is said to be **saturated** if it satisfies the following axiom.

MS5 for any morphism $f: X \rightarrow Y, g: W \rightarrow X$ and $h: V \rightarrow W$ such that $f \circ g$ and $g \circ h$ belong to \mathcal{S} , the morphism g belongs to \mathcal{S} . \square

Proposition 5.2. *Let \mathcal{S} be a left multiplicative system in a category \mathcal{C} . For any morphism $f: X \rightarrow Y$ in \mathcal{C} , $Q(f)$ is an isomorphism in $\mathcal{C}_{\mathcal{S}}$ if and only if there exist morphisms $g: W \rightarrow X, h: V \rightarrow W$ in \mathcal{C} such that $f \circ g \in \mathcal{S}$ and $g \circ h \in \mathcal{S}$.*

Proof. Let $f: X \rightarrow Y$ be a morphism in \mathcal{C} . Assume that $Q(f)$ is an isomorphism. Then by Lemma 4.5.(iv), there exists $g: W \rightarrow X$ in \mathcal{C} such that $f \circ g \in \mathcal{S}$. Since $Q(f \circ g) = Q(f) \circ Q(g)$ is an isomorphism, $Q(g) = Q(f)^{-1} \circ Q(f \circ g)$ is also an isomorphism. By Lemma 4.5.(iv) again, there exists $h: V \rightarrow W$ in \mathcal{C} such that $g \circ h \in \mathcal{S}$.

Conversely, assume that $f \circ g \in \mathcal{S}$ and $g \circ h \in \mathcal{S}$. Then $Q(f) \circ Q(g)$ and $Q(g) \circ Q(h)$ are isomorphisms. Hence $Q(g)$ has a left inverse and a right inverse, and thus $Q(g)$ is an isomorphism. Therefore $Q(f)$ is an isomorphism. \square

Corollary 5.3. *Let \mathcal{S} be a left multiplicative system in a category \mathcal{C} . Then the following conditions are equivalent.*

(i) \mathcal{S} is left saturated.

(ii) $\mathcal{S}' = \{f \in \text{Mor}(\mathcal{C}) \mid Q(f) \text{ is an isomorphism}\}.$

Proof. Set $\mathcal{S}' = \{f \in \text{Mor}(\mathcal{C}) \mid Q(f) \text{ is an isomorphism}\}.$

(i) \Rightarrow (ii). If $f \in \mathcal{S}$, then by Lemma 4.5.(iv), $Q(f)$ is an isomorphism, and thus $f \in \mathcal{S}'$. Conversely, if $(f: X \rightarrow Y) \in \mathcal{S}'$, then $Q(f)$ is an isomorphism. By Proposition 5.2, there exist morphisms $g: W \rightarrow X, h: V \rightarrow W$ in \mathcal{C} such that $f \circ g \in \mathcal{S}$ and $g \circ h \in \mathcal{S}$. Since \mathcal{S} is left saturated, we obtain $f \in \mathcal{S}$.

(ii) \Rightarrow (i). For any morphisms $f: X \rightarrow Y, g: W \rightarrow X$ and $h: V \rightarrow W$ such that $f \circ g \in \mathcal{S}$ and $g \circ h \in \mathcal{S}$, $Q(f)$ is an isomorphism by Proposition 5.2. Hence $f \in \mathcal{S}' = \mathcal{S}$. \square

In the same way, we have the following proposition and corollary.

Proposition 5.4. *Let \mathcal{S} be a right multiplicative system in a category \mathcal{C} . For any morphism $f: X \rightarrow Y$ in \mathcal{C} , $Q^r(f)$ is an isomorphism in $\mathcal{C}_{\mathcal{S}}^r$, if and only if there exist morphisms $g: Y \rightarrow Z, h: Z \rightarrow W$ in \mathcal{C} such that $g \circ f \in \mathcal{S}$ and $h \circ g \in \mathcal{S}$. \square*

Corollary 5.5. *Let \mathcal{S} be a right multiplicative system in a category \mathcal{C} . Then the following conditions are equivalent.*

(i) \mathcal{S} is right saturated.

(ii) $\mathcal{S} = \{f \in \text{Mor}(\mathcal{C}) \mid Q^r(f) \text{ is an isomorphism}\}.$

□

Proposition 5.6. *Let \mathcal{S} be a multiplicative system in a category \mathcal{C} . Then the following conditions are equivalent.*

(i) \mathcal{S} is left saturated.

(ii) \mathcal{S} is right saturated.

(iii) \mathcal{S} is saturated.

Proof. Assume that $f: X \rightarrow Y$, $g: W \rightarrow X$ and $h: V \rightarrow W$ are any morphisms such that $f \circ g \in \mathcal{S}$ and $g \circ h \in \mathcal{S}$. Then $Q(f \circ g) = Q(f) \circ Q(g)$ and $Q(g \circ h) = Q(g) \circ Q(h)$ are isomorphisms.

(i) \Rightarrow (iii). Since \mathcal{S} is left saturated, $f \in \mathcal{S}$ and thus $Q(f)$ is an isomorphism. Hence $Q(g)$ is an isomorphism. By Corollary 5.3, we obtain $g \in \mathcal{S}$.

(iii) \Rightarrow (i). Since \mathcal{S} is saturated, $g \in \mathcal{S}$ and thus $Q(g)$ is an isomorphism. Hence $Q(f)$ is an isomorphism. By Proposition 5.4, there exists $e: Y \rightarrow Z$ such that $e \circ f \in \mathcal{S}$. Since \mathcal{S} is saturated, we obtain $f \in \mathcal{S}$.

(ii) \Leftrightarrow (iii). Similarly, this follows from Corollary 5.5 and Proposition 5.2.

□

Acknowledgment. This work was supported by JSPS Grant-in-Aid for Scientific Research (C) Grant Number 22540043.

References

- [KS06] Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006. MR MR2182076 (2006k:18001)
- [OT13] Masahiro Ohno and Hiroyuki Terakawa, *A spectral sequence and nef vector bundles of the first Chern class two on hyperquadrics*, ANNALI DELL' UNIVERSITA' DI FER- RARA, Published online: 11 September 2013.
- [OT14] ———, *Triangulated Categories. I: Foundations*, The Tsuru University Review, No.79 (March, 2014), pp.1-15.
- [Ver77] Jean-Louis Verdier, *Categories dérivées. Quelques résultats (Etat O)*., Semin. Geom. algebr. Bois-Marie, SGA 4 1/2, Lect. Notes Math. 569, 262-311 (1977), 1977.

Received date : May, 7, 2014

Accepted date : June, 4, 2014