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Topological Classification of Nonsingular Tropical Affine Cubic Curves

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1. Introduction

Tropical geometry has been intensely studied over the last decade motivated by its connection with algebraic geometry, such as in [18]. On the other hand, it has wide applications to many fields of mathematics and other sciences since tropical geometry is a research of piecewise linear functions [2, 3, 9, 11, 17].

In this paper, we will classify tropical nonsingular plane cubic curves. We define tropical curves as the corner locus of tropical homogeneous polynomials in three variables embedded in a tropical projective surface $\mathbb{P}^{2,\mathrm{trop}}$. This definition is equivalent to saying that tropical curves are the dequantization of plane projective curves [4, 8, 16]. As in [1] and [8], we say a tropical curve is nonsingular if all the branch points are simplicial, namely, the point has exactly three edges and the dual triangle has the lattice area $1/2$.

The classification of tropical lines is rather trivial, and the classification of tropical plane quadratic curves is given by [20]. In [13], Kato obtained the classification of tropical plane cubic curves that are obtained as the dequantization of cubic curves not passing through the three poles of the projective plane. In this paper, we dequantize cubic curves passing through poles too.

The following is our main theorem.

Theorem 3.2. The set of homeomorphism classes of tropical nonsingular cubic curves has a one-to-one correspondence to Table B.

The theorem is a consequence of:

Proposition 3.1. The refinements of the subpolygons of a triangle of degree 3 are listed in Table A up to linear translations.

Every tropical cubic curve is dual to a refinement of a subpolygon of the triangle of degree 3 (see Proposition 2.1). Since every refinement given in Proposition 3.1 corresponds to a tropical cubic curve, we obtain the theorem.

2. Tropical Curves

Let K be Puiseux series field $\overline{\mathbb{C}((t))}$ and define the valuation $v_K : K^\times \to \mathbb{R}$ as

$$v_K : K^\times \ni \textstyle\sum_{i\in\mathbb{Q}} a_i t^i \mapsto \min\{i \mid a_i \neq 0\}.$$

We define $v_{K^n} : (K^\times)^n \ni (x_1, \ldots, x_n) \mapsto (v_K(x_1), \ldots, v_K(x_n)) \in \mathbb{R}^n$.

Let $F \in K[x_1, \ldots, x_n]$ be a nonzero polynomial. Denote the zero locus of F in the torus $(K^\times)^n$ by $Z(F)$, i.e.,

$$Z(F) = \{(a_1, \ldots, a_n) \in (K^\times)^n \mid F(a_1, \ldots, a_n) = 0\}.$$

We define tropical affine hypersurface $AH(F)$ to be the closure of $-v_K(Z(F))$.

From now on, we assume $n = 3$. For a polynomial $F = \sum_{i,j,k\geq 0} a_{ijk} x^i y^j z^k \in K[x, y, z]$, define f to be a function defined by

$$f : \mathbb{R}^3 \ni (X, Y, Z) \mapsto \max(-v_K(a_{ijk}) + iX + jY + kZ) \in \mathbb{R}.$$

Then $AH(F)$ equals the corner locus of f. See [12] or [19] for more details.

Remark 1. Let $\oplus, \otimes$ be binary operators on $\mathbb{R}$ defined by

$$a \oplus b = \max(a, b), \ a \otimes b = a + b.$$

Then $(\mathbb{R}, \oplus, \otimes)$ is a semifield without a zero element [2, 7, 10, 19]. We call this semifield a tropical semifield. Using these operators, we can rewrite the above function f as $f = \bigoplus_{i,j,k\geq 0} v_K(a_{ijk}) X^i Y^j Z^k$. Thus f is a polynomial function over the tropical semifield.

We say that two elements (a_1, a_2, a_3), (b_1, b_2, b_3) of $\mathbb{R}^3$ are equivalent if there exists a real number λ such that $a_i = b_i + \lambda$ for every $i \in \{1, 2, 3\}$. This is an equivalence relation and we define the tropical projective plane $\mathbb{P}^{2,\mathrm{trop}}$ to be $\mathbb{P}^{2,\mathrm{trop}} = \mathbb{R}^3/\sim$. This is homeomorphic to the interior of a triangle.

Remark 2. $\mathbb{P}^{2,\mathrm{trop}}$ can be obtained naturally by gluing three $\mathbb{R}^2$'s.

Remark 3. Tropical semifield is sometimes defined as $(\mathbb{T}, \oplus, \otimes)$, where $\mathbb{T}$ is the union of $\mathbb{R}$ and a zero element $-\infty$. In this setting, $\mathbb{P}^{2,\mathrm{trop}}$ will be defined as $(\mathbb{T}^3 \setminus \{(-\infty, -\infty, -\infty)\})/\sim$, which is homeomorphic to a triangle with the boundaries. Three vertices are $(-\infty : -\infty : 0), (-\infty : 0 : -\infty)$, and $(0 : -\infty : -\infty)$.

Let $F \in K[x, y, z]$ be a homogeneous polynomial of degree d. Since the set $AH(F)$ is closed under the relation $\sim$, $AH(F)$ defines a subset of $\mathbb{P}^{2,\mathrm{trop}}$. We call this subset (projective) tropical curve of degree d and write as $V(F)$. Tropical curves are finite graphs with unbounded edges.

Denote the coefficient of the term $x^i y^j z^k$ of F by $a_{i,j,k}$. Let $\tilde{\Delta}_F$ be a subset of $\mathbb{R}^3$ defined as follows:

$$\tilde{\Delta}_F = \mathrm{ConvexHull}\{(i, j, \alpha) \in \mathbb{R}^3 \mid a_{i,j,d-i-j} \neq 0, \ \alpha \leq v_K(a_{i,j,d-i-j})\}.$$

Let S_F be the set of all the edges of $\tilde{\Delta}_F$. Put π to be the canonical projection $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. We denote the refinement of the Newton polytope $\Delta_F = \pi(\tilde{\Delta}_F)$ with the boundary being $\pi(S_F)$ as Subdiv_F. The following proposition is well-known [20].

Proposition 2.1. Subdiv_F and $V(F)$ are dual to each other. Each vertex (resp. edge, face) of Subdiv_F corresponds to a face (resp. an edge, a point) in $V(F)$ with the adjacency relations preserved.

Example 1. For $G = txy + xz + yz + tz^2$, Δ_G is a quadrilateral with vertices being $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ as shown in Figure 1. Since vertices of $\tilde{\Delta}_G$ are $(0,0,-1)$, $(0,1,0)$, $(1,0,0)$ and $(1,1,-1)$, Subdiv_G consists of two triangles glued together along an edge (the dot lines). By drawing the dual line of Subdiv_G, we obtain $V(G)$ (the thin lines).

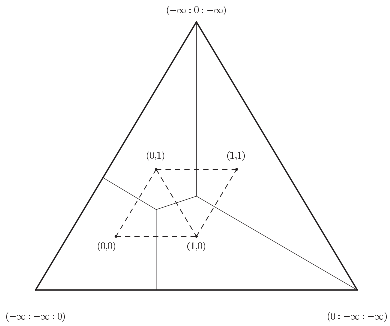


Figure 1: $V(G)$ and Subdiv_G

We say that $V(F)$ is nonsingular if Subdiv_F refines Δ_F maximally. That is to say, every refined subpolygon is a triangle of area $1/2$. Note that all the vertices of a nonsingular tropical curve are of degree 3.

Remark 4. Every singular tropical curve is a degeneration of nonsingular tropical curves [8]. For example, the singular tropical curve $V(xy + xz + yz + z^2)$ (Figure 2) is the $r \to +0$ limit of a nonsingular tropical curve $V(t^r xy + xz + yz + t^r z^2)$ (Figure 3).

3. Classification of Nonsingular Tropical Cubic Curves

Thanks to the duality between Subdiv_F and $V(F)$, we can reduce the problem of classifying tropical cubic curves to the problem of classifying the refinement of a subpolygon of the triangle of degree 3.

Proposition 3.1. The refinements of the subpolygons of the triangle of degree 3 are listed in Table A up to linear translations and permutations of three vertices. Each refinement is realizable as Subdiv_F for some F.

Proof. Shown by enumeration. The polynomial written below realizes each refinement. □

We used a Maple package [6] to find concrete tropical cubic polynomials.

Theorem 3.2. The set of homeomorphism classes of nonsingular tropical cubic curves has a one-to-one correspondence to Table B.

Proof. Follows from Proposition 2.1 and Proposition 3.1. □

Corollary 3.3. There are 73 nonsingular tropical cubic curves up to homeomorphism.

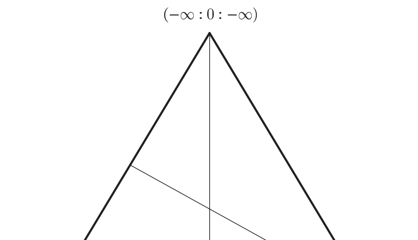


Figure 2: singular curve

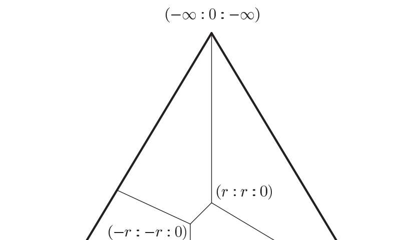
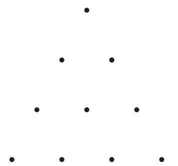
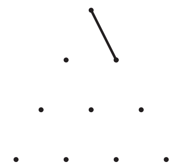
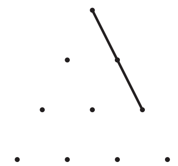
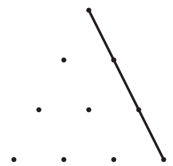
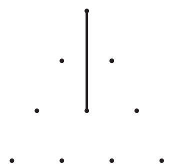
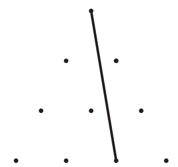
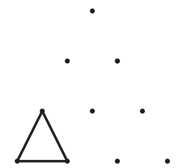
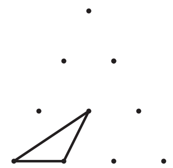
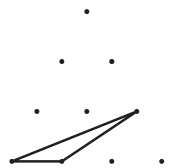
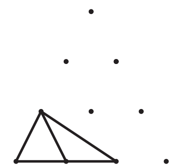
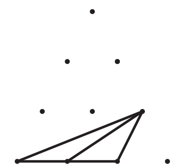
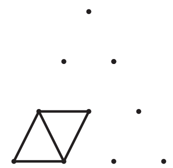
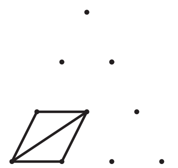
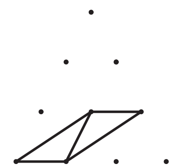
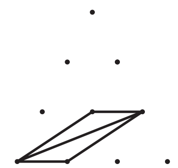
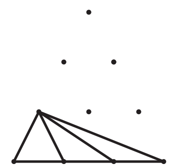
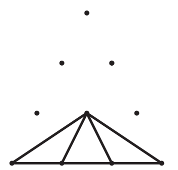
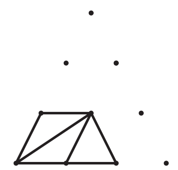
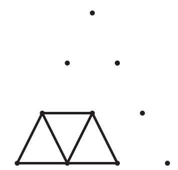
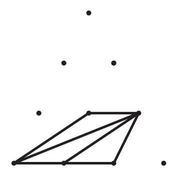
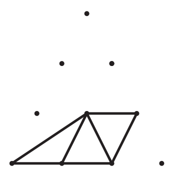
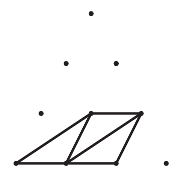
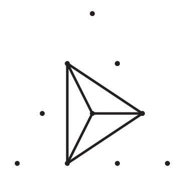
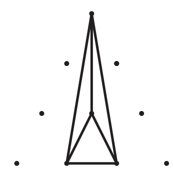
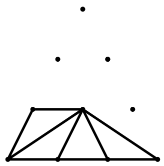
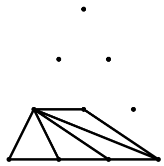
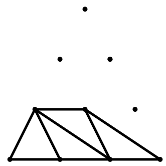
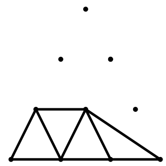
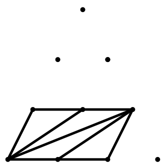
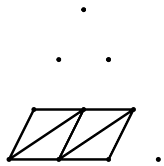
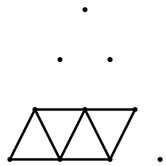
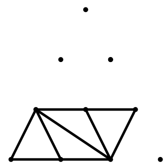
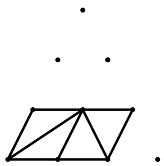
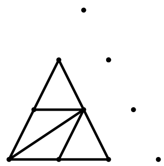
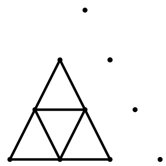
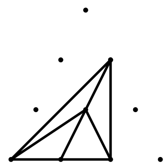
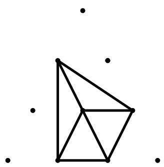
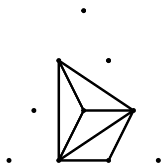
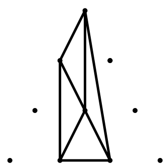
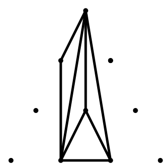
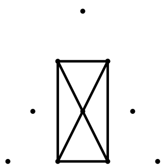
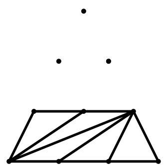
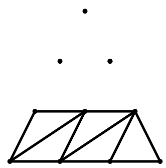
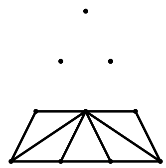
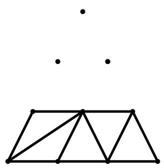
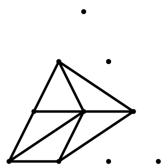
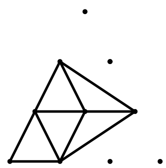
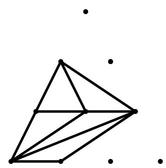
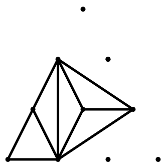
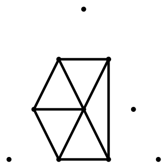
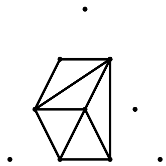
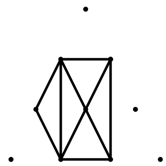
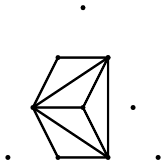
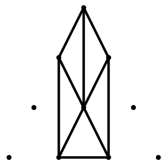
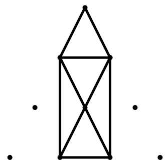
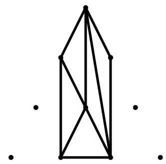
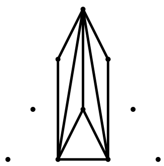
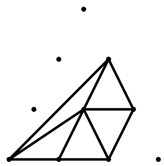
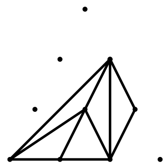
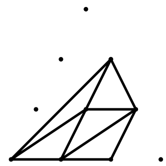
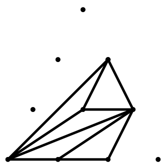
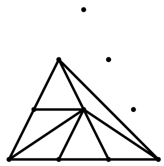
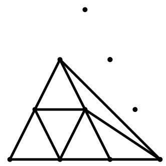
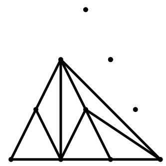
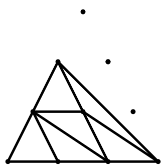
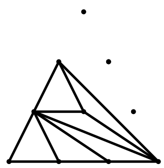
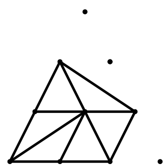
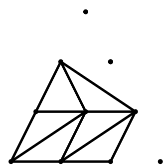
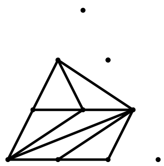
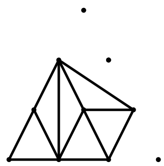
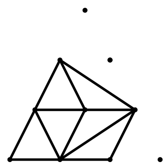
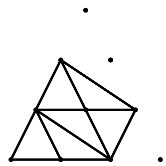


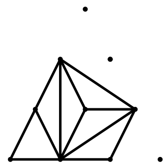
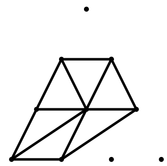
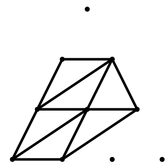
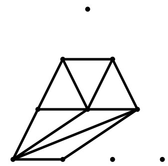
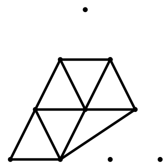
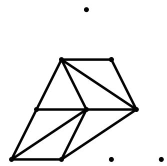
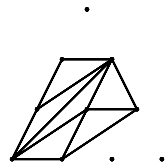
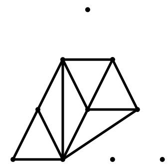
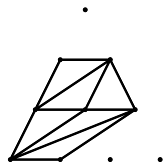
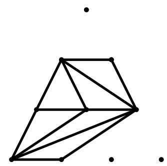
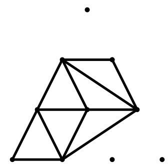
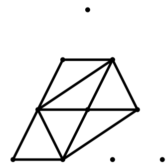
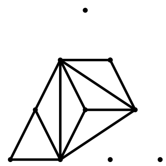
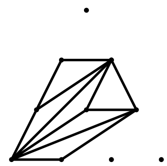
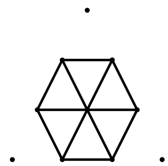
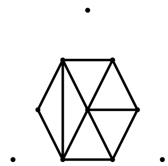
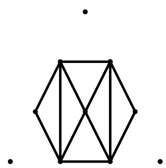
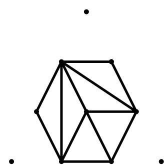
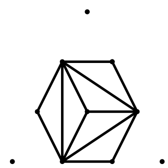
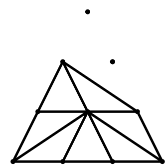
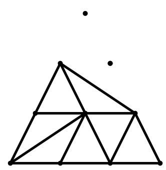
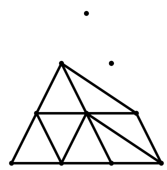
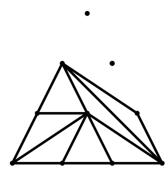
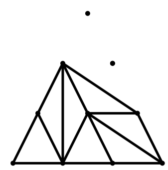
Figure 3: nonsingular curve

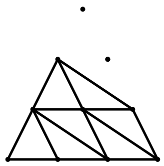
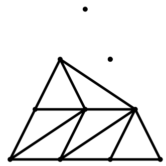
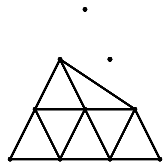
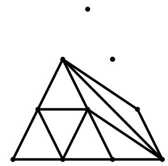
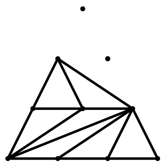
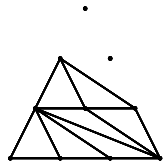
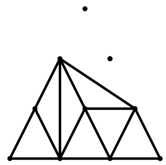
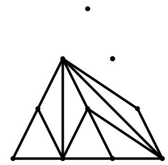
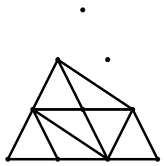
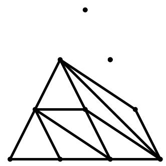
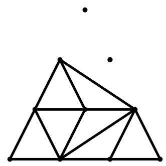
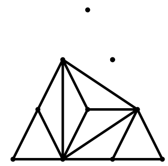
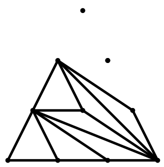
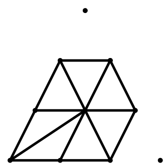
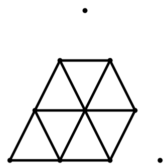
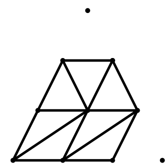
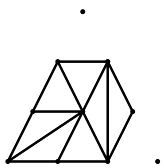
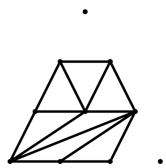
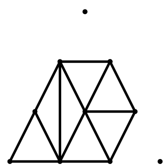
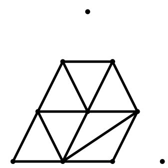
Table A. List of Subdiv$_F$

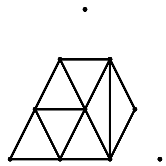
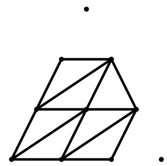
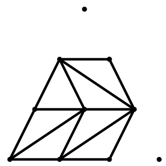
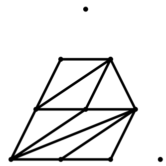
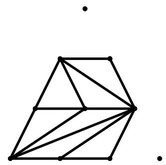
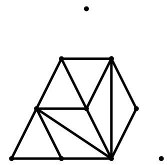
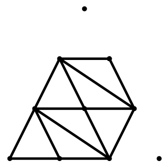
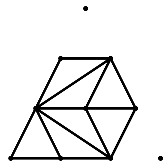
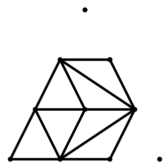
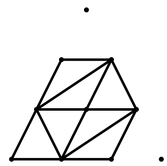
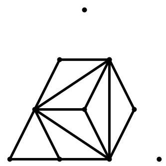
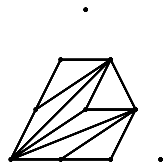
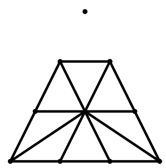
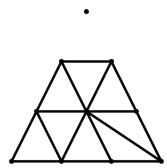
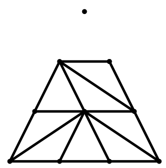
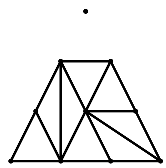
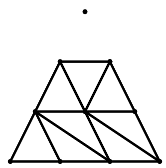
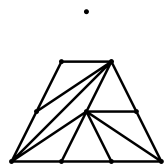
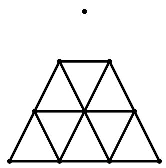
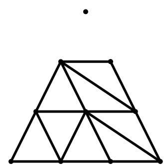
 a cubic monomial	 xy^2+y^3	 $tx^2y+xy^2+ty^3$	 $tx^3+x^2y+xy^2+ty^3$
 $xyz+y^3$	 x^2z+y^3	 $xz^2+yz^2+z^3$	 $xyz+xz^2+z^3$
 $x^2y+xz^2+z^3$	 $tx^2z+xz^2+yz^2+z^3$	 $x^2y+tx^2z+xz^2+z^3$	 $txyz+xz^2+yz^2+tz^3$
 $xyz+txz^2+tyz^2+z^3$	 $tx^2y+xyz+xz^2+tz^3$	 $x^2y+txyz+txz^2+z^3$	 $tx^3+x^2z+xz^2+yz^2+tz^3$
 $tx^3+x^2z+xyz+xz^2+tz^3$	 $tx^2z+xyz+xz^2+tyz^2+z^3$	 $tx^2z+xyz+xz^2+yz^2+tz^3$	 $x^2y+tx^2z+txyz+xz^2+z^3$
 $tx^2y+x^2z+xyz+xz^2+tz^3$	 $x^2y+tx^2z+xyz+xz^2+tz^3$	 $tx^2y+xyz+ty^2z+txz^2$	 $ty^3+tx^2z+xyz+txz^2$

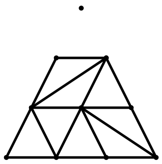
 $t^3x^3 + tx^2z + xyz$ $+ xz^2 + t^3yz^2 + tz^3$	 $tx^3 + x^2z + t^5xyz$ $+ txz^2 + t^3yz^2 + t^3z^3$	 $t^3x^3 + x^2z + txyz$ $+ txz^2 + yz^2 + t^3z^3$	 $t^3x^3 + tx^2z + xyz$ $+ xz^2 + tyz^2 + t^3z^3$
 $x^2y + t^3x^2z + txyz$ $+ txz^2 + t^3yz^2 + z^3$	 $tx^2y + t^3x^2z + xyz$ $+ xz^2 + t^3yz^2 + tz^3$	 $t^3x^2y + tx^2z + xyz$ $+ xz^2 + tyz^2 + t^3z^3$	 $t^3x^2y + x^2z + txyz$ $+ txz^2 + yz^2 + t^3z^3$
 $t^3x^2y + tx^2z + txyz$ $+ xz^2 + t^3yz^2 + tz^3$	 $t^3x^2z + xyz + t^3y^2z$ $+ txz^2 + tyz^2 + z^3$	 $tx^2z + xyz + ty^2z$ $+ xz^2 + yz^2 + tz^3$	 $txy^2 + tx^2z + xyz + xz^2 + tz^3$
 $tx^2y + tx^2z + xyz$ $+ ty^2z + txz^2$	 $tx^2y + t^3x^2z + xyz$ $+ ty^2z + txz^2$	 $ty^3 + tx^2z + xyz$ $+ ty^2z + txz^2$	 $ty^3 + tx^2z + xyz$ $+ t^3y^2z + txz^2$
 $txy^2 + tx^2z + xyz$ $+ ty^2z + txz^2$	 $t^3x^3 + x^2y + tx^2z$ $+ txyz + xz^2 + t^3yz^2 + z^3$	 $t^3x^3 + x^2y + tx^2z$ $+ xyz + xz^2 + t^3yz^2 + tz^3$	 $tx^3 + t^2x^2y + x^2z$ $+ xyz + xz^2 + t^2yz^2 + tz^3$
 $t^3x^3 + tx^2y + x^2z$ $+ xyz + xz^2 + t^2yz^2 + tz^3$	 $t^2x^2y + xyz + t^2y^2z$ $+ txz^2 + yz^2 + z^3$	 $tx^2y + xyz + ty^2z$ $+ txz^2 + tyz^2 + t^3z^3$	 $tx^2y + xyz + t^2y^2z$ $+ t^3xz^2 + tyz^2 + tz^3$

 $tx^2y+xyz+ty^2z$ $+txz^2+t^3yz^2+t^6z^3$	 txy^2+tx^2z+xyz $+ty^2z+txz^2+tyz^2$	 txy^2+tx^2z+xyz $+t^3y^2z+txz^2+tyz^2$	 txy^2+tx^2z+xyz $+ty^2z+txz^2+t^3yz^2$
 txy^2+tx^2z+xyz $+t^3y^2z+t^3xz^2+tyz^2$	 $txy^2+ty^3+tx^2z$ $+xyz+ty^2z+txz^2$	 $txy^2+t^3y^3+tx^2z$ $+xyz+ty^2z+txz^2$	 $t^3xy^2+ty^3+tx^2z$ $+xyz+ty^2z+txz^2$
 $t^3xy^2+ty^3+tx^2z$ $+xyz+t^3y^2z+txz^2$	 $t^2x^2y+t^2xy^2+tx^2z$ $+xyz+xz^2+z^3$	 $t^3x^2y+txy^2+tx^2z$ $+xyz+xz^2+z^3$	 $x^2y+txy^2+tx^2z$ $+xyz+xz^2+tz^3$
 $x^2y+t^3xy^2+t^3x^2z$ $+xyz+txz^2+z^3$	 $t^3x^3+t^2x^2z+xyz$ $+t^3y^2z+t^2xz^2$ $+t^2yz^2+t^3z^3$	 $t^3x^3+tx^2z+xyz$ $+ty^2z+xz^2+yz^2+tz^3$	 $tx^3+x^2z+xyz+ty^2z$ $+xz^2+t^2yz^2+t^4z^3$
 tx^3+x^2z+xyz $+ty^2z+txz^2+yz^2+t^3z^3$	 $x^3+x^2z+txyz$ $+t^4y^2z+txz^2+yz^2+t^3z^3$	 $t^2x^2y+t^2x^2z+xyz$ $+t^2y^2z+t^2xz^2+t^2yz^2+t^3z^3$	 $tx^2y+t^3x^2z+xyz$ $+t^3y^2z+txz^2+t^2yz^2+t^2z^3$
 $x^2y+t^3x^2z+xyz$ $+t^3y^2z+txz^2+tyz^2+z^3$	 tx^2y+tx^2z+xyz $+ty^2z+txz^2+t^3yz^2+t^6z^3$	 $txy^2+t^3x^2z+xyz$ $+ty^2z+txz^2+tyz^2+t^3z^3$	 tx^2y+tx^2z+xyz $+ty^2z+t^2xz^2+yz^2+t^4z^3$

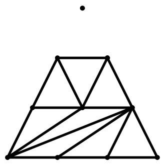
 $tx^2y + t^3x^2z + xyz + y^2z + xz^2 + tyz^2 + t^3z^3$	 $t^2x^2y + t^2xy^2 + xyz + ty^2z + t^2xz^2 + yz^2 + tz^3$	 $t^3x^2y + txy^2 + xyz + t^3y^2z + txz^2 + tyz^2 + tz^3$	 $tx^2y + xy^2 + xyz + y^2z + t^5xz^2 + tyz^2 + t^3z^3$
 $tx^2y + txy^2 + xyz + ty^2z + txz^2 + tyz^2 + t^3z^3$	 $t^4x^2y + t^9xy^2 + xyz + t^4y^2z + t^3xz^2 + t^4yz^2 + t^6z^3$	 $t^2x^2y + xy^2 + xyz + t^3y^2z + txz^2 + tyz^2 + z^3$	 $tx^2y + txy^2 + xyz + ty^2z + xz^2 + t^2yz^2 + t^4z^3$
 $tx^2y + txy^2 + xyz + t^3y^2z + t^3xz^2 + tyz^2 + tz^3$	 $tx^2y + t^3xy^2 + xyz + ty^2z + t^3xz^2 + yz^2 + tz^3$	 $tx^2y + t^3xy^2 + xyz + ty^2z + txz^2 + tyz^2 + t^3z^3$	 $tx^2y + txy^2 + xyz + t^3y^2z + txz^2 + tyz^2 + t^3z^3$
 $tx^2y + t^4xy^2 + xyz + ty^2z + xz^2 + t^2yz^2 + t^4z^3$	 $tx^2y + xy^2 + xyz + t^4y^2z + t^4xz^2 + t^2yz^2 + tz^3$	 $tx^2y + txy^2 + tx^2z + xyz + ty^2z + txz^2 + tyz^2$	 $tx^2y + txy^2 + tx^2z + xyz + ty^2z + txz^2 + t^3yz^2$
 $t^3x^2y + txy^2 + tx^2z + xyz + ty^2z + txz^2 + t^3yz^2$	 $tx^2y + t^3xy^2 + tx^2z + xyz + ty^2z + txz^2 + t^3yz^2$	 $tx^2y + t^3xy^2 + t^3x^2z + xyz + ty^2z + txz^2 + t^3yz^2$	 $t^3x^3 + t^3x^2y + t^2x^2z + xyz + t^3y^2z + t^2xz^2 + t^2yz^2 + t^3z^3$
 $t^4x^3 + t^2x^2y + tx^2z + xyz + t^3y^2z + xz^2 + tyz^2 + z^3$	 $tx^3 + t^2x^2y + x^2z + xyz + t^2y^2z + txz^2 + t^2yz^2 + t^4z^3$	 $t^3x^3 + t^7x^2y + t^2x^2z + xyz + t^3y^2z + t^2xz^2 + t^2yz^2 + t^3z^3$	 $t^3x^3 + t^4x^2y + tx^2z + txyz + t^3y^2z + xz^2 + t^3yz^2 + t^4z^3$

 $tx^3+t^2x^2y+x^2z+xyz+t^2y^2z+txz^2+yz^2+t^3z^3$	 $t^5x^3+tx^2y+t^2x^2z+xyz+t^3y^2z+xz^2+tyz^2+z^3$	 $t^3x^3+tx^2y+tx^2z+xyz+ty^2z+txz^2+tyz^2+t^3z^3$	 $tx^3+t^4x^2y+x^2z+xyz+t^2y^2z+txz^2+t^2yz^2+t^4z^3$
 $t^6x^3+x^2y+t^3x^2z+xyz+t^3y^2z+txz^2+tyz^2+z^3$	 $t^2x^3+t^3x^2y+t^2x^2z+txyz+t^3y^2z+t^3xz^2+yz^2+t^5z^3$	 $t^4x^3+t^2x^2y+tx^2z+xyz+ty^2z+xz^2+t^2yz^2+t^4z^3$	 $t^3x^3+t^5x^2y+tx^2z+xyz+ty^2z+xz^2+t^2yz^2+t^4z^3$
 $t^4x^3+t^2x^2y+tx^2z+xyz+ty^2z+t^2xz^2+yz^2+t^4z^3$	 $t^2x^3+t^4x^2y+x^2z+xyz+ty^2z+t^2xz^2+tyz^2+t^5z^3$	 $t^4x^3+x^2y+t^2x^2z+xyz+ty^2z+txz^2+tyz^2+t^4z^3$	 $t^6x^3+tx^2y+t^3x^2z+xyz+ty^2z+txz^2+t^3yz^2+t^6z^3$
 $t^2x^3+t^4x^2y+t^2x^2z+txyz+t^2y^2z+t^3xz^2+yz^2+t^5z^3$	 $t^3x^2y+t^3xy^2+t^3x^2z+xyz+t^3y^2z+txz^2+tyz^2+z^3$	 $tx^2y+txy^2+tx^2z+xyz+ty^2z+txz^2+tyz^2+t^3z^3$	 $tx^2y+t^2xy^2+t^3x^2z+xyz+t^2y^2z+txz^2+tyz^2+tz^3$
 $t^4x^2y+txy^2+t^2x^2z+xyz+t^2y^2z+t^2xz^2+t^2yz^2+t^3z^3$	 $x^2y+t^4xy^2+t^3x^2z+xyz+t^5y^2z+txz^2+t^2yz^2+z^3$	 $tx^2y+txy^2+tx^2z+xyz+ty^2z+txz^2+t^3yz^2+t^6z^3$	 $tx^2y+txy^2+t^3x^2z+xyz+ty^2z+txz^2+tyz^2+t^3z^3$

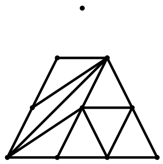
 $t^3x^2y+txy^2+tx^2z+xyz+ty^2z+txz^2+tyz^2+t^3z^3$	 $tx^2y+txy^2+t^3x^2z+xyz+t^3y^2z+txz^2+tyz^2+tz^3$	 $t^2x^2y+t^4xy^2+t^4x^2z+xyz+ty^2z+txz^2+yz^2+z^3$	 $tx^2y+txy^2+t^5x^2z+xyz+t^4y^2z+t^2xz^2+yz^2+z^3$
 $tx^2y+t^4xy^2+t^5x^2z+xyz+ty^2z+t^2xz^2+yz^2+z^3$	 $t^3x^2y+t^2xy^2+x^2z+xyz+ty^2z+txz^2+yz^2+t^3z^3$	 $tx^2y+t^4xy^2+tx^2z+xyz+y^2z+t^2xz^2+yz^2+t^4z^3$	 $tx^2y+txy^2+tx^2z+xyz+t^4y^2z+t^2xz^2+yz^2+t^4z^3$
 $tx^2y+t^3xy^2+t^3x^2z+xyz+ty^2z+txz^2+tyz^2+t^3z^3$	 $tx^2y+txy^2+t^3x^2z+xyz+t^3y^2z+txz^2+tyz^2+t^3z^3$	 $t^4x^2y+txy^2+tx^2z+xyz+t^4y^2z+t^2xz^2+yz^2+t^4z^3$	 $tx^2y+txy^2+t^6x^2z+xyz+t^6y^2z+t^3xz^2+t^3yz^2+tz^3$
 $t^3x^3+t^2x^2y+t^3xy^2+t^2x^2z+xyz+t^3y^2z+t^2xz^2+t^2yz^2+t^3z^3$	 $t^3x^3+t^3x^2y+t^4xy^2+tx^2z+xyz+t^3y^2z+xz^2+yz^2+tz^3$	 $x^3+tx^2y+t^4xy^2+x^2z+xyz+t^2y^2z+txz^2+t^2yz^2+t^4z^3$	 $tx^3+t^2x^2y+t^4xy^2+x^2z+xyz+t^3y^2z+xz^2+t^4yz^2+t^6z^3$
 $t^3x^3+t^3x^2y+t^4xy^2+tx^2z+xyz+t^3y^2z+t^2xz^2+yz^2+t^4z^3$	 $t^3x^3+t^2x^2y+t^2xy^2+t^2x^2z+xyz+t^{10}y^2z+t^2xz^2+t^6yz^2+t^3z^3$	 $t^3x^3+tx^2y+txy^2+tx^2z+xyz+ty^2z+txz^2+tyz^2+t^3z^3$	 $t^3x^3+t^2x^2y+t^4xy^2+t^2x^2z+xyz+ty^2z+t^2xz^2+t^2yz^2+t^5z^3$



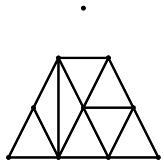
$$\begin{aligned} &tx^3 + tx^2y + t^2xy^2 \\ &\quad + tx^2z + xyz + t^5y^2z \\ &\qquad + t^2xz^2 + t^2yz^2 + t^5z^3 \end{aligned}$$



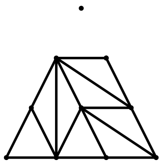
$$\begin{aligned} &t^5x^3 + x^2y + t^7xy^2 \\ &\quad + t^3x^2z + t^2xyz + t^{10}y^2z \\ &\qquad + t^3xz^2 + t^6yz^2 + t^4z^3 \end{aligned}$$



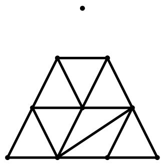
$$\begin{aligned} &tx^3 + x^2y + txy^2 \\ &\quad + x^2z + xyz + t^8y^2z \\ &\qquad + txz^2 + t^5yz^2 + t^3z^3 \end{aligned}$$



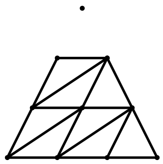
$$\begin{aligned} &t^6x^3 + t^4x^2y + t^5xy^2 \\ &\quad + t^2x^2z + txyz + t^4y^2z \\ &\qquad + xz^2 + t^4yz^2 + t^5z^3 \end{aligned}$$



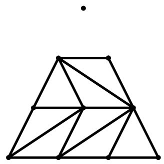
$$\begin{aligned} &t^3x^3 + t^3x^2y + t^6xy^2 \\ &\quad + x^2z + xyz + ty^2z \\ &\qquad + xz^2 + t^2yz^2 + t^4z^3 \end{aligned}$$



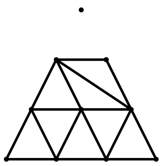
$$\begin{aligned} &t^3x^3 + x^2y + t^2xy^2 \\ &\quad + tx^2z + xyz + t^4y^2z \\ &\qquad + xz^2 + t^3yz^2 + t^5z^3 \end{aligned}$$



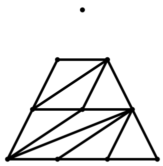
$$\begin{aligned} &t^3x^3 + x^2y + t^2xy^2 \\ &\quad + tx^2z + xyz + t^6y^2z \\ &\qquad + xz^2 + t^2yz^2 + tz^3 \end{aligned}$$



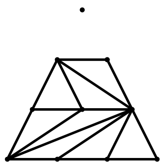
$$\begin{aligned} &t^4x^3 + tx^2y + t^7xy^2 \\ &\quad + tx^2z + txyz + t^6y^2z \\ &\qquad + xz^2 + t^3yz^2 + tz^3 \end{aligned}$$



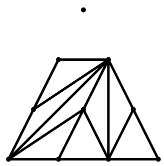
$$\begin{aligned} &t^4x^3 + t^2x^2y + t^4xy^2 \\ &\quad + tx^2z + xyz + ty^2z \\ &\qquad + xz^2 + yz^2 + t^3z^3 \end{aligned}$$



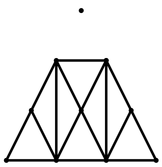
$$\begin{aligned} &t^3x^3 + x^2y + t^6xy^2 \\ &\quad + tx^2z + t^2xyz + t^{10}y^2z \\ &\qquad + txz^2 + t^5yz^2 + t^2z^3 \end{aligned}$$



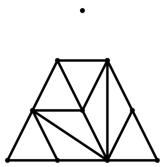
$$\begin{aligned} &t^5x^3 + x^2y + t^5xy^2 \\ &\quad + t^3x^2z + txyz + t^5y^2z \\ &\qquad + t^2xz^2 + t^3yz^2 + t^2z^3 \end{aligned}$$



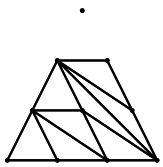
$$\begin{aligned} &t^9x^3 + t^4x^2y + xy^2 \\ &\quad + t^3x^2z + xyz + t^3y^2z \\ &\qquad + txz^2 + tyz^2 + z^3 \end{aligned}$$



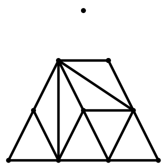
$$\begin{aligned} &t^4x^3 + t^2x^2y + txy^2 \\ &\quad + x^2z + xyz + ty^2z \\ &\qquad + xz^2 + t^2yz^2 + t^4z^3 \end{aligned}$$



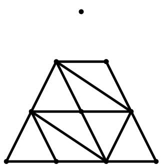
$$\begin{aligned} &t^6x^3 + t^4x^2y + t^4xy^2 \\ &\quad + x^2z + txyz + t^3y^2z \\ &\qquad + t^2xz^2 + t^2yz^2 + t^6z^3 \end{aligned}$$



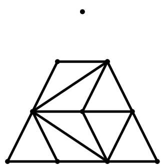
$$\begin{aligned} &t^2x^3 + t^5x^2y + t^9xy^2 \\ &\quad + x^2z + txyz + t^3y^2z \\ &\qquad + xz^2 + yz^2 + t^2z^3 \end{aligned}$$



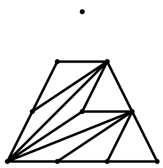
$$\begin{aligned} &t^4x^3 + tx^2y + t^4xy^2 \\ &\quad + tx^2z + xyz + ty^2z \\ &\qquad + txz^2 + t^3yz^2 + t^6z^3 \end{aligned}$$



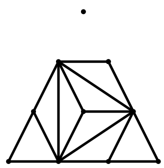
$$\begin{aligned} &t^3x^3 + tx^2y + t^3xy^2 \\ &\quad + tx^2z + xyz + ty^2z \\ &\qquad + t^3xz^2 + tyz^2 + t^6z^3 \end{aligned}$$



$$\begin{aligned} &t^3x^3 + tx^2y + txy^2 \\ &\quad + tx^2z + xyz + t^3y^2z \\ &\qquad + t^3xz^2 + tyz^2 + t^6z^3 \end{aligned}$$



$$\begin{aligned} &t^6x^3 + x^2y + txy^2 \\ &\quad + t^4x^2z + txyz + t^8y^2z \\ &\qquad + t^3xz^2 + t^5yz^2 + t^3z^3 \end{aligned}$$



$$\begin{aligned} &t^6x^3 + tx^2y + t^3xy^2 \\ &\quad + t^3x^2z + xyz + ty^2z \\ &\qquad + txz^2 + t^3yz^2 + t^6z^3 \end{aligned}$$

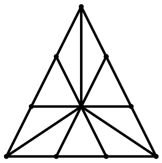
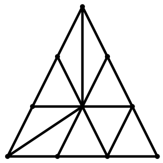
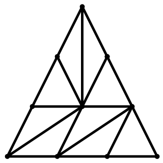
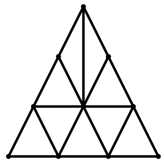
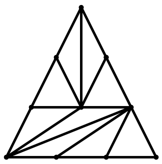
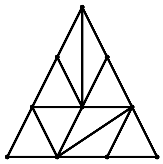
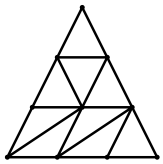
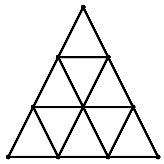
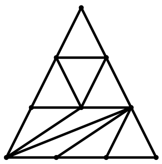
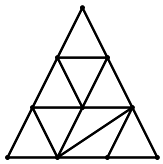
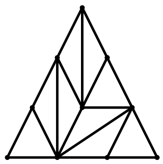
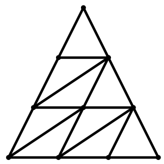
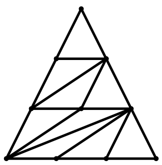
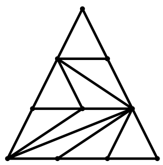
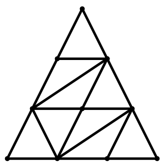
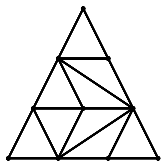
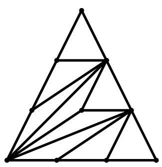
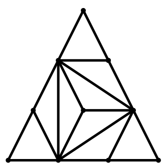
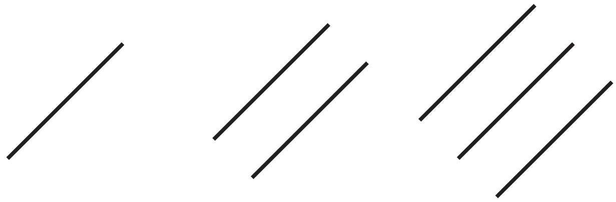
 $t^3x^3 + t^2x^2y + t^2xy^2 + t^3y^3 + t^2x^2z + xyz + t^2y^2z + t^2xz^2 + t^2yz^2 + t^3z^3$	 $t^4x^3 + tx^2y + t^3xy^2 + t^6y^3 + tx^2z + xyz + t^4y^2z + t^2xz^2 + t^3yz^2 + t^4z^3$	 $t^9x^3 + t^2x^2y + t^2xy^2 + t^3y^3 + t^5x^2z + xyz + t^2y^2z + t^2xz^2 + t^2yz^2 + t^3z^3$	 $t^5x^3 + t^2x^2y + t^2xy^2 + t^3y^3 + t^2x^2z + xyz + t^2y^2z + t^2xz^2 + t^2yz^2 + t^5z^3$
 $t^{17}x^3 + t^3x^2y + t^3xy^2 + t^4y^3 + t^{11}x^2z + xyz + t^2y^2z + t^6xz^2 + tyz^2 + t^2z^3$	 $t^8x^3 + t^2x^2y + t^2xy^2 + t^3y^3 + t^4x^2z + xyz + t^2y^2z + txz^2 + t^2yz^2 + t^4z^3$	 $t^8x^3 + t^2x^2y + t^2xy^2 + t^3y^3 + t^4x^2z + xyz + ty^2z + txz^2 + yz^2 + z^3$	 $t^5x^3 + t^2x^2y + t^2xy^2 + t^5y^3 + t^2x^2z + xyz + t^2y^2z + t^2xz^2 + t^2yz^2 + t^5z^3$
 $t^{11}x^3 + tx^2y + txy^2 + t^3y^3 + t^7x^2z + xyz + ty^2z + t^4xz^2 + tyz^2 + t^2z^3$	 $t^6x^3 + tx^2y + txy^2 + t^3y^3 + t^3x^2z + xyz + ty^2z + txz^2 + tyz^2 + t^3z^3$	 $t^7x^3 + t^2x^2y + t^2xy^2 + t^3y^3 + t^3x^2z + xyz + t^2y^2z + xz^2 + t^3yz^2 + z^5$	 $t^{11}x^3 + t^6x^2y + t^6xy^2 + t^{11}y^3 + t^6x^2z + t^3xyz + t^6y^2z + t^2xz^2 + t^2yz^2 + z^3$
 $t^6x^3 + x^2y + t^2xy^2 + t^8y^3 + t^3x^2z + xyz + t^4y^2z + txz^2 + tyz^2 + z^3$	 $t^8x^3 + x^2y + t^3xy^2 + t^7y^3 + t^5x^2z + xyz + t^2y^2z + t^3xz^2 + tyz^2 + t^2z^3$	 $t^6x^3 + tx^2y + txy^2 + t^6y^3 + t^3x^2z + xyz + t^3y^2z + txz^2 + tyz^2 + t^3z^5$	 $t^6x^3 + tx^2y + t^3xy^2 + t^6y^3 + t^3x^2z + xyz + ty^2z + txz^2 + tyz^2 + t^3z^3$
 $t^{10}x^3 + tx^2y + txy^2 + t^{10}y^3 + t^6x^2z + xyz + t^6y^2z + t^3xz^2 + t^3yz^2 + tz^3$	 $t^6x^3 + tx^2y + t^3xy^2 + t^6y^3 + t^3x^2z + xyz + ty^2z + txz^2 + t^3yz^2 + t^6z^3$		

Table B. List of tropical cubic curves
($K =$ the number of branch points)

$\underline{K = 0}$

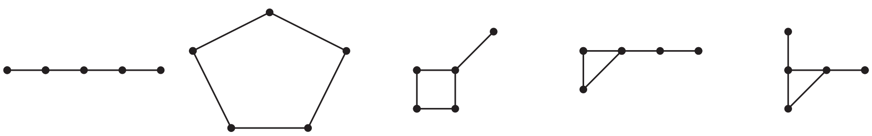
$\emptyset$



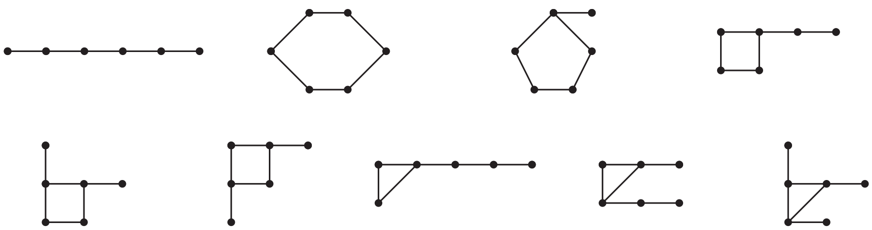
$\underline{K = 1}$　　$\underline{K = 2}$　　$\underline{K = 3}$

$\underline{K = 4}$

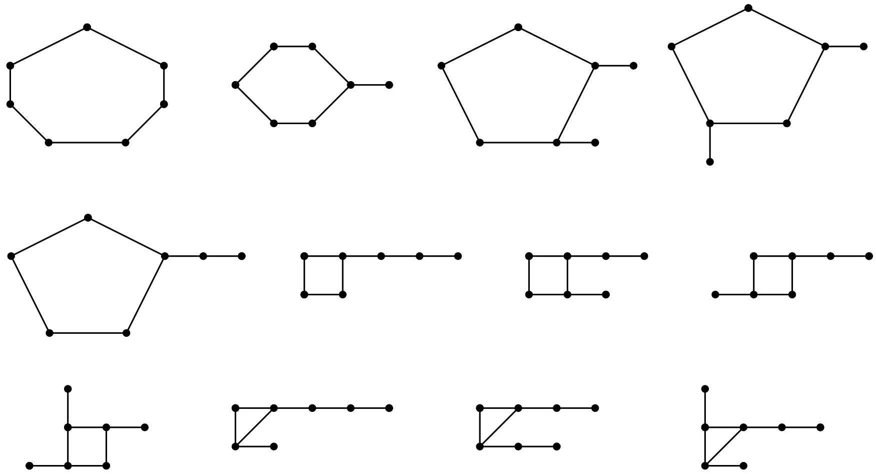
$\underline{K = 5}$



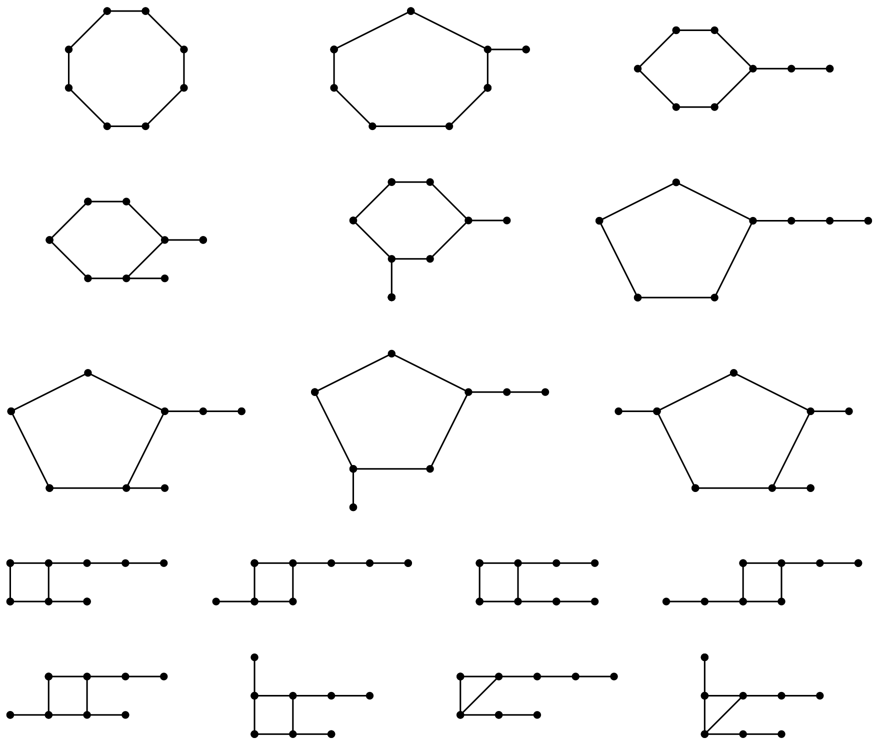
$\underline{K = 6}$



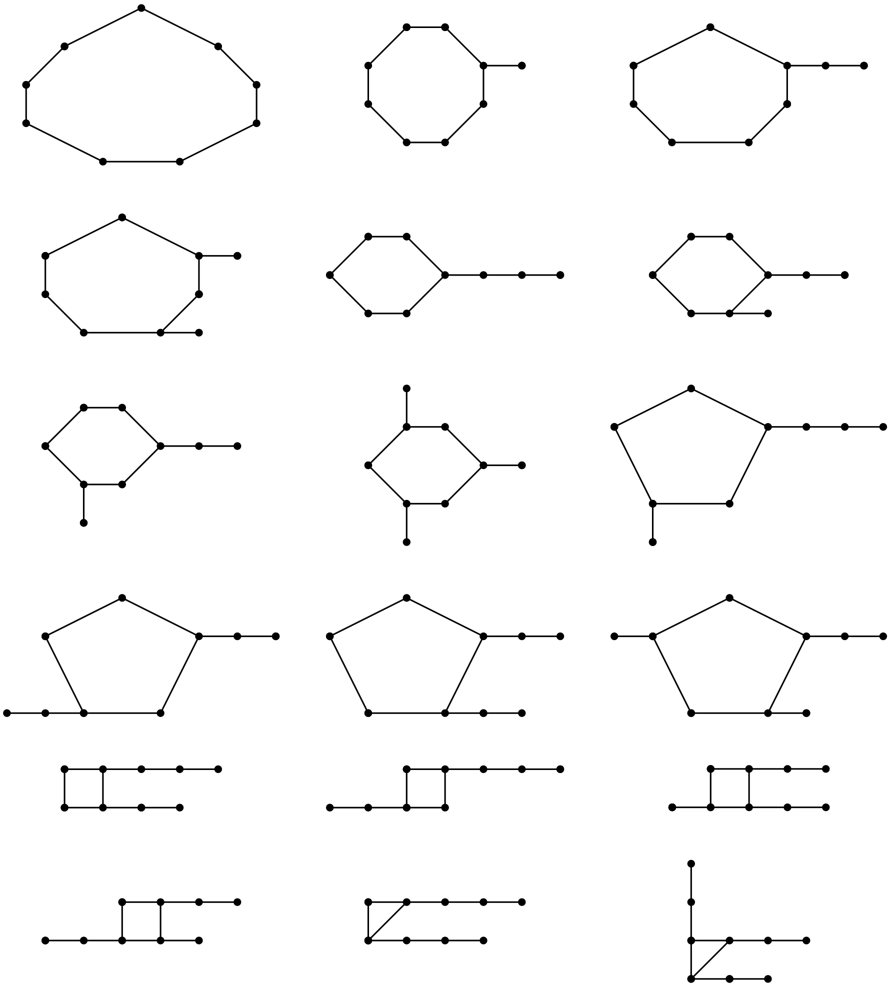
$\underline{K = 7}$



$\underline{K = 8}$



$\underline{K = 9}$



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